Second Order Problems

Parabolic Case

Parabolic Problems

- heat equation
- fundamental solution
- initial-boundary value problem
- maximum principle
- product setup
- representation by series
- Crank-Nickolson method
- method of lines

Heat equation

We concentrate on the following parabolic PDE

$$u_t(x,t) = \Delta u(x,t) + r(x,t)$$

with $x \in \mathbb{R}$, $x \in \mathbb{R}^2$ or $x \in \mathbb{R}^3$.

At time $t = t_0 =$ we pose a single initial condition

$$u(x,t_0) = u_0(x) \quad \forall x\,.$$

Problems may be posed on the whole space $\mathbb{R}^d$ or on a domain $\Omega \subset \mathbb{R}^d$. In the latter case, boundary conditions on $\partial\Omega$ need to be added.

Solution on the whole space

The considered equation describes the diffusive transport of matter as well as the conduction of heat.
Without any source term, a concentrated amount of mass or energy will be (normal-) distributed over the whole space with a variance that grows with time.
In the case of $x \in \mathbb{R}$, it holds

$$u(x,t) = E(x,t) = \frac{1}{2\sqrt{\pi t}} \exp\left(-\frac{x^2}{4t}\right)$$

It is easy to check, that this expression fulfills the PDE, but it is less easy to prove that the integral is constant and equal to 1. At $x = 0$, where the quantity was initially concentrated, there is the maximum of $u(\cdot,t)$ for all $t > 0$. At $t = 0$ we have a singularity (Dirac's δ-'function').

Remark

We call the function $E(x,t)$ the fundamental solution of the heat equation.

Visualization

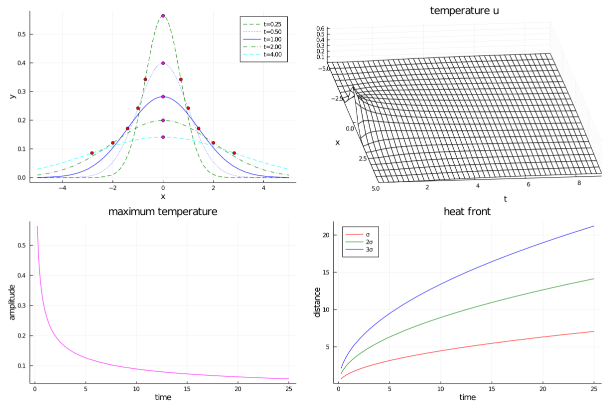

Fig. 19: Fundamental solution to heat equation

Solution in n-dimensions

Theorem

For a given bounded and continuous function $u_0 : \mathbb{R}^n \to \mathbb{R}$, the unique solution to the Cauchy problem for the homogeneous heat equation with initial state u_0 is given by

$$u_{hom}(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \int_{\mathbb{R}^n} u_0(y) \exp\left(-\frac{|x-y|^2}{4t}\right) dy$$

Remark

We denote here by $|\cdot|$ the Euclidean distance in $\mathbb{R}^n$.

Evolution in 2d

Here we follow the solution corresponding to a concentrated heat distribution at the origin $x_1 = x_2 = 0$ for $t = 0$.

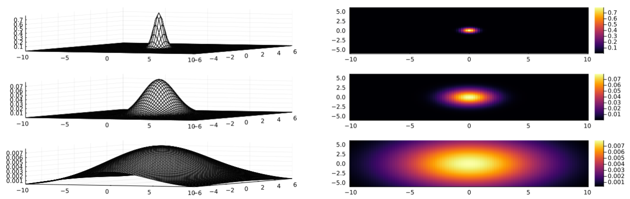

Fig. 20: Sequence of solutions at $t = 0.1$, $t = 1.0$ and $t = 10$

Source term

In the presence of a source term $r(x,t)$, but with zero initial conditions, the solution can also be obtained in terms of the fundamental solution $E(x,t)$

$$u_{src}(x,t) = \int_{\tau=0}^{t} \int_{\mathbb{R}^n} r(\zeta,\tau) E(x-\zeta, t-\tau)\, d\zeta\, d\tau\,.$$

Remark

Given the linearity of the PDE, the superposition principle holds, hence

$$u(x,t) = u_{hom}(x,t) + u_{src}(x,t)$$

is the unique solution to the complete Cauchy problem for the heat equation with non-trivial initial conditions and non-vanishing source term.

Evaluation

Remark

It has to be mentioned that the evaluation of the integral in the homogeneous problem and the double integral in the non-homogeneous case is, in general, quite challenging.
This is particularly so in 2 or 3 space dimensions.

Remark

Variable coefficients lead to a more general heat equation

$$c(x,t)u_t(x,t) = \mathrm{div}\,(\kappa(x,t)\,\mathrm{grad}\, u)$$

with heat capacity c and conductivity κ.

Nonlinearities may appear if $c = c(x,t,u)$, and/or $\kappa = \kappa(x,t,u)$.

Neither case can be treated by the above formulas.

Bounded domains

Mostly, modeling is focused on a certain domain $\Omega \subset \mathbb{R}^n$.
Given a positive T, we denote by Z the space-time cylinder

$$\overline{\Omega} \times [0, T] =: Z\,,$$

and by

$$\partial Z = \partial\Omega \times [0, T] \cup \Omega \times \{0\}$$

the parabolic boundary of Z.

Remark

The parabolic boundary is not the topological boundary of the $(n+1)$-dimensional set Z.

It is the surface of the cylinder Z without the cover.

It is the set where initial conditions or boundary conditions are posed.

Maximum principle

Theorem (maximum principle)

Let $T > 0$ and $u \in \mathcal{C}^2(\Omega \times (0, T)) \cap \mathcal{C}^2(\overline{\Omega} \times [0, T])$ be a smooth solution to the homogeneous heat equation

$$u_t(x, z) = \Delta u(x, t) \quad \text{in int } Z\,,$$

then

$$\min_{\partial Z} u \le \min_{Z} u \le \max_{Z} u \le \max_{\partial Z} u\,.$$

Corollary

If there exists a solution to an initial-boundary value problem for the heat equation on Z, it is unique.

Fourier series approach

Now, we construct solutions to the heat equation

$$u_t(x, t) = u_{xx}(x, t)$$

in the 1d case with $\Omega = (0, 1)$.
Assume vanishing Dirichlet boundary conditions at the ends of the interval, $x = 0$ and $x = 1$, i.e., $u(0, t) = u(1, t)$ for all t.
The initial condition reads

$$u(x, 0) = \varphi(x) \quad \text{for } x \in (0, 1)$$

with a continuous function φ.
It turns out that one can find solutions in the form of products $u(x, t) = v(x)w(t)$.

Remark

Due to the linearity of the PDE, solutions may be superposed.

Eigenforms

The PDE applied to a product yields conditions on each factor

$$v''(x) + \lambda v(x) = 0\,, \quad \text{where } v(0) = v(1) = 0$$

and

$$\dot{w}(t) + \lambda w(t) = 0$$

with the common factor $\lambda \in \mathbb{R}$.
The problem for v has an infinite spectrum of eigensolutions in the form

$$v_k(x) = \sin(k\pi x)\,, \quad k \in \mathbb{N}\,.$$

From this, we obtain $\lambda_k = k^2\pi^2$ and hence

$$w_k(t) = \exp(-k^2\pi^2 t)\,.$$

Series

Using linearity, we compose the solution by combining modes

$$u(x, t) = \sum_{k=1}^{\infty} c_k v_k(x) w_k(t) = \sum_{k=1}^{\infty} c_k \sin(k\pi x) \exp(-k^2\pi^2 t)$$

with constant coefficients c_k.
Using the initial conditions at $t = 0$, which renders the time-dependent factor equal to 1, we conclude that the initial displacement

$$\varphi(x) = u(x, 0) = \sum_{k=1}^{\infty} c_k \sin(k\pi x)$$

defines the amplitudes c_k via its Fourier sine expansion.

Example

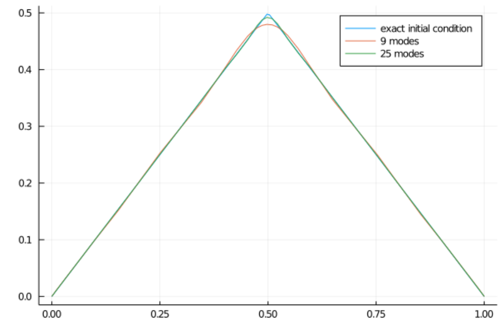

Fig. 21: Fourier sine expansion of triangular distribution

Solution

Theorem

Let φ be continuous, its derivative φ' be piecewise continuous, and let compatibility conditions $\varphi(0) = 0 = \varphi(1)$ be satisfied.
Then the series

$$u(x, t) = \sum_{k=1}^{\infty} c_k \sin(k\pi x) \exp(-k^2\pi^2 t)$$

with

$$c_k = 2\int_0^1 \varphi(x) \sin(k\pi x)\, dx$$

is the uniquely defined solution to the initial boundary problem for the considered heat equation.

Properties

Corollary

For $t > 0$, the solution has continuous derivatives of any order, i.e., the solution is smooth.

Remark

The elements of the spectrum are called modes of the problem, with increasing index k, they have a growing frequency and number of zeros.
The higher k, the faster the time-dependent factor decays.
For this reason, one is mostly interested in the first, basic modes.

Remark

There are two differences to the wave equation:

- *the time dependence is exponential, not sinosoidal*
- *the coefficient is squared*

Thus, we have quick convergence to zero instead of oscillation.

Example

Example

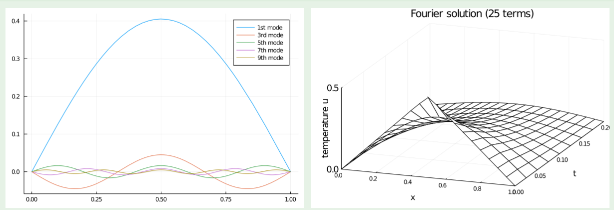

Fig. 22: Fourier solution for triangular initial distribution