Second Order Problems

Hyperbolic Case

Hyperbolic Problems

- Wave equation
- d'Alembert formula
- solution in 3d
- solution in 2d
- initial-boundary-value problem

Wave equation

We consider the equation

$$u_{tt}(x,t) = \Delta u(x,t) \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n,$$

where the Laplace operator sums up the second order pure x-derivatives.

Remark

In hyperbolic and parabolic problems, one of the variables is distinguished, we speak about the time variable, the remaining variables are called space variables.
Speaking about a one-dimensional problem, we usually have in mind a problem with one space variable, while in fact, it is a problem on $\mathbb{R}^2$.

We complete the wave equation by initial conditions on $u(\cdot, t_0)$ and $u_t(\cdot, t_0)$, which are initial displacement and speed, respectively.

One-dimensional case

With $x \in \mathbb{R}$, the wave equation becomes

$$u_{tt}(x,t) = c^2 u_{xx}(x,t),$$

with the wave speed c.
In new variables $\zeta = x - c\zeta$, $\eta = x + c\zeta$, this becomes

$$w_{\zeta\eta}(\zeta,\eta) = 0,$$

From this, it follows that the solution has the form

$$u(x,t) = F(\zeta) + G(\eta) = F(x - ct) + G(x + ct),$$

Initial conditions

Let's assume $t_0 = 0$ and $u(x,0) = u_0(x)$, $u_t(x,0) = v_0(x)$.
It follows

$$F(x) + G(x) = u_0(x), \quad -cF'(x) + cG'(x) = v_0(x).$$

Taking the derivative, we obtain a system for F' and G',

$$\begin{aligned} F' + G' &= u_0', \\ -cF' + cG' &= v_0 \end{aligned}$$

with the solution

$$G' = \frac{cu_0' + v_0}{2c}, \quad F' = -\frac{cu_0' - v_0}{2c},$$

from which F and G can be obtained by integration.

d'Alembert's formula

Theorem

Let $u_0 \in C^2(\mathbb{R})$ and $v_0 \in C^1(\mathbb{R})$. Then the unique solution to the Cauchy problem

$$u_{tt} = u_{xx}, \quad u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x)$$

is

$$u(x,t) = \frac{u_0(x-t) + u_0(x+t)}{2} + \frac{1}{2}\int_{x-t}^{x+t} v_0(\xi)\, d\xi.$$

Remark

In mathematics, a transformation to dimensionless quantities is preferred in order to eliminate constants like c in this case.
As an exercise, one can derive the formula in the general case as well.

Some solutions

Example

We solve the wave equation $u_{tt} = u_{xx}$ on $\mathbb{R}$ with triangular initial conditions for the displacement ($u_0(x) = \max(0, a|x - 0.5|)$) and/or a cosine ($v_0(x) = b\cos(2x)$)/rectangular pulse ($v_0(x) = \chi_{[-2.5,2.5]}(x)$) for the velocity.

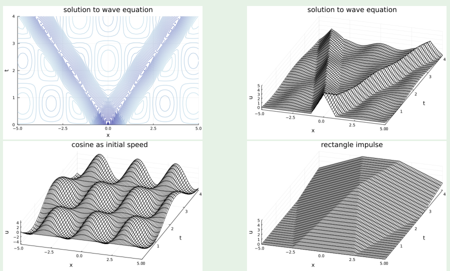

Fig. 13: d'Alembert's solution

Reformulation

The scalar wave equation can be reformulated as a system of two first order PDEs. To this end, we introduce the quantities $v = u_t$ and $\epsilon = u_x$. Their physical meanings are velocity and deformation, respectively.
They are governed by the equations

$$\begin{aligned} \epsilon_t &= v_x \\ v_t &= \epsilon_x. \end{aligned}$$

The second of the two equations is the original equation in the new variables, while the first one ensures geometrical compatibility.
It expresses the Schwarz equality of second order mixed derivatives of the displacement function $u(x,t)$.

Hyperbolic systems

Remark

In general, a vector equation of the form

$$u_t(x,t) + A(x,t)u_x(x,t) = G(x,t,u)$$

with a matrix A, that has a full system of eigenvectors, with all eigenvalues real, is called a hyperbolic system of PDEs.
If all eigenvalues are different, one speaks of strong hyperbolicity.

Remark

Hyperbolic balance laws of the form

$$u_t + \mathrm{div}\, F(x,t,u) = G(x,t,u)$$

are of special interest in continuum physics.
Note that the flux tensor F does not depend on derivatives, it is a function of the state variables comprised in the vector u alone.

Solution

For the wave equation, the matrix A becomes

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

with the eigenvalues -1 and $+1$.
Transformation of the state vector $u = (u_1, u_2)^T$ to the eigenbasis gives

$$\partial_t(u_1 + u_2) + \partial_x(u_1 + u_2) = 0\,, \quad \partial_t(u_1 - u_2) - \partial_x(u_1 - u_2) = 0\,.$$

The problem splits into two independent first order Cauchy problems for the functions $z_1 = u_1 + u_2$ and $z_2 = u_1 - u_2$.

Superposition

We consider once more the triangular initial displacement as in the previous example.

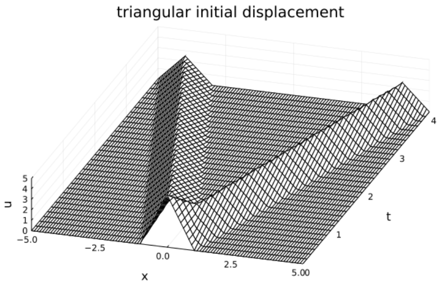

Fig. 14: Two waves corresponding to two components of vector $z = (z_1, z_2)^T$

Source terms

If we add an inhomogeneity on the right-hand side of the wave equation, in the case of vanishing initial conditions for u and u_t, the solution becomes

$$u(x,t) = \frac{1}{2} \int_0^t \int_{x-t+\tau}^{x+t-\tau} f(\xi,\tau)\, d\xi\, d\tau\,.$$

In the special case $f(x,t) = f(t)$, the solution u is also independent of x, and it is a second anti-derivative of f.

The 3d case

Theorem (Kirchhoff)

The unique solution to the problem

$$\begin{aligned} u_{tt}(x,t) &= \Delta u(x,t)\,, \\ u(x,0) &= 0\,, \\ u_t(x,0) &= p(x)\,, \end{aligned}$$

has the form

$$u(x,t) = \frac{1}{4\pi t} \int_{S_t(x)} p(y)\, dy\,,$$

where the integral is calculated with respect to the surface measure on the sphere.

Remark

The right-hand side of the Kirchhoff formula is the integral mean value of all initial values in the distance t (times wave speed in general) from the considered point x.

Sketch of proof

Proof: We calculate the right-hand side of the formula in spherical coordinates φ, ϑ, representing $y = x + t\vec{n}$, where $\vec{n}$ is the unit outer vector of the sphere $S_t(x)$

$$\vec{n} = (\sin(\vartheta)\cos(\varphi), \sin(\vartheta)\sin(\varphi), \cos(\vartheta))^T\,.$$

At $t = 0$ we confirm the initial conditions, and for $t > 0$ we derive

$$\begin{aligned} u(x,t) &= \frac{t}{4\pi} \int_0^{\pi} \int_0^{2\pi} p(x + t\vec{n}(\vartheta,\varphi)) \sin(\vartheta)\, d\varphi\, d\vartheta\,, \\ u_t(x,t) &= \frac{1}{t} u + \frac{1}{4\pi t} \int_{S_t(x)} \nabla p(y) \vec{n}(y)\, dy = \frac{1}{t} u + \frac{1}{4\pi t} \int_{B_t(x)} \nabla \cdot (\nabla p(y))\, dy\,, \\ u_{tt}(x,t) &= \frac{1}{4\pi t} \frac{d}{dt} \int_0^t \int_{S_t(x)} \nabla \cdot (\nabla p(y))\, dy = \Delta u(x,t)\,. \end{aligned}$$

A lemma

Lemma

The solution to

$$\begin{aligned} u_{tt}(x,t) &= \Delta u(x,t)\,, \\ u(x,0) &= p(x)\,, \\ u_t(x,0) &= 0\,, \end{aligned}$$

is given by

$$u(x,t) = \frac{\partial}{\partial t} \left(\frac{1}{4\pi t} \int_{S_t(x)} p(y)\, dy \right)\,.$$

Remark

Swapping the conditions for u and u_t results in a time derivative, otherwise there is no difference to Kirchhoff's formula.

Proof: If $u(\cdot,\cdot)$ satisfies the wave equations, $v(\cdot,\cdot) = u_t(\cdot,\cdot)$ fulfills it as well, furthermore, it meets initial conditions for its values, while $v_t(x,0) = u_{tt}(x,0) = 0$.

Combination

Now, due to linearity, the two right-hand side terms may be added to find a solution to

$$\begin{aligned} u_{tt}(x,t) &= \Delta u(x,t)\,, \\ u(x,0) &= u_0(x)\,, \\ u_t(x,0) &= v_0(x)\,, \end{aligned}$$

which leads to

$$u(x,t) = \frac{1}{4\pi t} \int_{S_t(x)} v_0(y)\, dy + \frac{\partial}{\partial t} \left(\frac{1}{4\pi t} \int_{S_t(x)} u_0(y)\, dy \right)\,.$$

Theorem

Assuming $u_0 \in \mathcal{C}^3(\mathbb{R}^3)$ and $v_0 \in \mathcal{C}^2(\mathbb{R}^3)$, the above formula gives the solution to the Cauchy problem for the 3d wave equation.

Descent

In two dimensions, the solution is obtained from the 3d case, where one assumes that $u(x_1, x_2, x_3, t)$ does not depend on x_3.
Of course, this requires that also the initial data must not contain any x_3.

Hence, the solution becomes

$$u(x,t) = \frac{1}{2\pi} \int_{B_t(x)} \frac{v_0(y)}{\sqrt{t^2 - |y-x|^2}}\, dy + \frac{\partial}{\partial t} \left(\frac{1}{2\pi} \int_{B_t(x)} \frac{u_0(y)}{\sqrt{t^2 - |y-x|^2}}\, dy \right)\,.$$

Notice that this time, integration is over the whole disc in $\mathbb{R}^2$, not just the circumference.

Remark

In the 3d case, an initial signal arrives at (x,t) exactly if its source $(y,0)$ is in a spatial distance equal to t (in general, multiplied by the wave speed c for $u_{tt} = c^2 u_{xx}$).
In 2d, also signals from a shorter distance may arrive – they originate from non-zero x_3-layers of 3d space.

Fourier series approach

Now, we construct solutions to the wave equation

$$u_{tt}(x,t) = u_{xx}(x,t)$$

in the 1d case with $\Omega = (0,1)$.
Further, we assume vanishing Dirichlet boundary conditions at the ends of the interval, $x = 0$ and $x = 1$, i.e.,
$u(0,t) = u_t(0,t) = u(1,t) = u_t(1,t) = 0$ for all t.
Initial condition for position u and speed $v = u_t$ are

$$u(x,0) = u_0(x)\,, \quad u_t(x,0) = v_0(x) \quad \text{for } x \in (0,1)$$

with continuous functions u_0 and v_0.

It turns out that one can find solutions in the form of products $u(x,t) = w(x)z(t)$.

Remark

Due to the linearity of the PDE, solutions may be superposed.

Eigenforms

The PDE applied to a product yields conditions on each factor

$$w''(x) + \lambda w(x) = 0\,, \quad \text{where } w(0) = w(1) = 0$$

and

$$\ddot{z}(t) + \lambda z(t) = 0$$

with the common factor $\lambda \in \mathbb{R}$.
The problem for w has an infinite spectrum of eigensolutions in the form

$$w_k(x) = \sin(k\pi x)\,, \quad k \in \mathbb{N}\,.$$

From this, we obtain $\lambda_k = k^2\pi^2$ and hence

$$z_k(t) = a_k \sin(k\pi t) + b_k \cos(k\pi t)\,.$$

Series

Using linearity, we compose the solution by combining modes

$$u(x,t) = \sum_{k=1}^{\infty} \sin(k\pi x)\left(a_k \cos(k\pi t) + b_k \sin(k\pi t)\right)$$

with free coefficients a_k and b_k.
Taking the time derivative at $t = 0$, comparison with the initial conditions, a_k and b_k are to be defined by Fourier series expansions

$$u_0(x) = u(x,0) = \sum_{k=1}^{\infty} a_k \sin(k\pi x)\,,$$

$$v_0(x) = u_t(x,0) = \sum_{k=1}^{\infty} k\pi b_k \sin(k\pi x)\,.$$

Solution

Theorem

Let u_0 and v_0 be continuous functions with piecewise continuous derivatives and zero boundary values.
With

$$a_k = 2\int_0^1 u_0(x)\sin(k\pi x)\,dx\,,$$

$$b_k = \frac{2}{k\pi}\int_0^1 v_0(x)\sin(k\pi x)\,dx\,,$$

the solution of the initial-boundary value problem to the wave equation is given by the series

$$u(x,t) = \sum_{k=1}^{\infty} \sin(k\pi x)\left(a_k \cos(k\pi t) + b_k \sin(k\pi t)\right)\,.$$

Example

Example

We solve the wave equation $u_{tt} = u_{xx}$ on $(0,1)$ with zero boundary conditions.
Let $u_0 = |x - 0.5|$ and $v_0 = 4x(1-x)$.

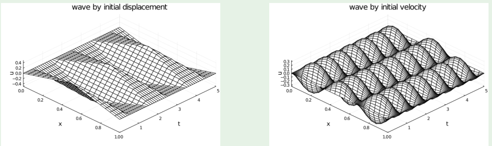

Fig. 15: Fourier solution – contributions of u_0 and v_0

Solution to the example problem

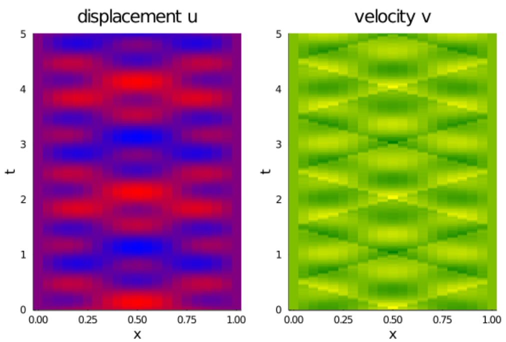

Fig. 16: Displacements and velocities u and $v = u_t$

Numerical approach to the wave equation

We consider the one-dimensional partial differential equation

$$u_{tt}(x,t) = c^2 u_{xx}(x,t)\,, \quad x \in (x_l, x_r)\,, \quad t \in [0,T]\,,$$

assuming at $t = 0$

$$u(x,0) = u_0(x)\,, \quad x \in [x_l, x_r]\,,$$

$$u_t(x,0) = v_0(x)\,, \quad x \in [x_l, x_r]\,.$$

and at the ends of the interval Dirichlet conditions

$$u(x_l,t) = u_l(t)\,, \quad u(x_r,t) = u_r(t), t \in [0,T]\,.$$

Possible generalizations are material inhomogeneity, damping and forcing, i.e. lower order terms, variable coefficients and a non-zero right-hand side.

Fourier method

As for analytical solutions, we can apply the decomposition method also in the case of a discrete, numerical approach.
We construct numerical solutions on an equidistant grid in the space domain, $x_j = x_0 + j \cdot h$.
We represent the right-hand side of the PDE, e.g. u_{xx}, by a matrix multiplication of the nodal values, here

$$c^2 u_{xx}(x_j) \approx AU$$

with $U = [u(x_j), j = 1, 2, \ldots, n]^T$.
Next, we perform an eigen-analysis of the matrix A, writing it as

$$A = Q\Lambda Q^{-1}$$

where the columns of Q are the eigenvectors, the corresponding eigenvalues λ are the entries of the diagonal matrix Λ.

Initial conditions

The initial conditions u_0 and v_0 are represented by their grid values U_0, resp. V_0, and both vectors are decomposed in the eigenbasis definded by Q.
The solution can be found in the form:

$$U(t) = \sum_{k=1}^{n} \left(a_k \cos(\lambda_k t) + b_k \sin(\lambda_k t)\right) Q(:,k)\,,$$

where the a_k and $\lambda_k b_k$ are the coefficients of U_0, V_0 in the basis Q,

$$Qa = U_0\,, \quad Q\Lambda b = V_0\,.$$

Remark

The approach works as long as the right-hand-side is linear in u and the matrix A has a full set of real eigenvectors.

Inhomogeneous material

Example

We solve a wave equation $u_{tt} = c^2 u_{xx}$ on $[-\pi, +\pi]$ with a variable coefficient $c^2 = 2 + \cos(7x)$ and a concentrated initial condition for the displacement $u_0(x) = \exp(-9(1+x)^2)$ with zero initial speed and zero boundary conditions.

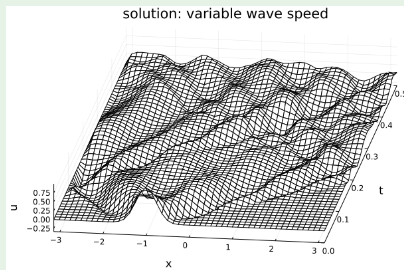

Fig. 17: Numerical solution

In the hyperbolic case, we can also represent the second order PDE by a system of two first order PDEs.
Next, we apply the well-known schemes, like Lax-Friedrich or Lax-Wendroff, to both components of the vector quantity. Finally, we compose the original function back from the new variables.

Remark (upwind)

In the case of the wave equation, we observe two waves, one of them running forward, one backward.
If we want to apply an upwind discretization, we first need to decouple the system, bringing it to a diagonal form. Then, to the component with positive speed, we apply backward differentiation, to the other one (with negative speed), we use forward differentiation.

Remark

As in the case of scalar balance laws, explicit difference schemes require a limit in the time-step to be stable, and hence to converge.
Again, the CFL condition is to be satisfied for all wave speeds of the hyperbolic problem. No wave may travel farther than of space step during one time step.

Example

We solve a hyperbolic system, which is equivalent to a wave equation equation, $u_t = 2v_x$, $v_t = 8u_x$, with a triangular initial condition for both the displacement and the speed, $u_0(x) = v_0(x) = \max(0, 1 - |1 + x|)$

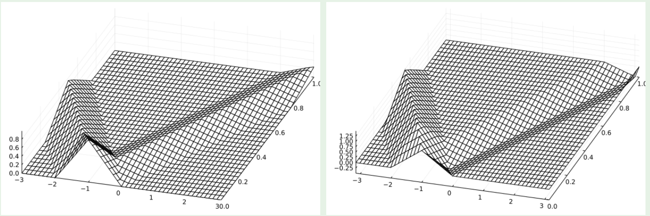

Fig. 18: LF solution

In the context of parabolic problems, we will encounter the method of lines (MoL), which may be also applied to hyperbolic problems.
Since this approach can easily use implicit ODE-solvers, it is also suitable to avoid stability problems.