Second Order Problems

Elliptic Case

Elliptic Problems

- Laplace equation
- harmonic functions
- Dirichlet problem
- Green's formula
- mean value theorem
- maximum principle
- uniqueness of solution
- weak formulation
- solution on circle
- Finite Difference Method
- Finite Element Method

Elliptic Problems

We consider the Laplace equation

$$\Delta u = u_{x_1x_1} + u_{x_2x_2} + \dots u_{x_nx_n} = 0 .$$

Solutions are called harmonic functions.

Remark

Mostly, we are concerned with two- or three-dimensional problems, and we rename the variables to x, y, z.

Remark

For PDEs on a disk or a sphere, it may be more convenient to represent the independent variables by polar or spherical coordinates.

Transformations

Theorem (invariance)

The operator Δ remains invariant under shifts and orthogonal transformations of coordinate systems, e.g. rotations.

If $x = x_0 + Q\zeta$, $u(x) = w(x(\zeta))$, then

$$\Delta u(x) = w_{\zeta_1\zeta_1} + w_{\zeta_2\zeta_2} + \dots w_{\zeta_n\zeta_n} = \Delta w(\zeta) .$$

Theorem (polar coordinates)

If we use $x = r\cos(\vartheta)$, $y = r\sin(\vartheta)$, the Laplace operator is given by

$$\Delta u(x, y) = r^{-1}(rw_r)_r + r^{-2}w_{\vartheta\vartheta} .$$

Examples of solutions

The equation $\Delta u(x) = 0$, $x \in \mathbb{R}^n$, has many obvious solutions, such as

- functions that are linear in each variable, e.g. $u(x) = \prod_{i=1}^{n} x_i$,
- in particular, linear functions $u(x) = c^T x$, with a constant vector of coefficients $c \in \mathbb{R}^n$,
- real and imaginary parts of holomorph complex functions (two-dimensional case).

Complex functions

Let

$$\Phi(z) = \Phi(x + y\mathrm{i}) = \varphi(x, y) + \mathrm{i}\psi(x, y)$$

with a differentiable function Φ, i.e., for all z there exists the limit

$$\phi'(z) = \lim_{h\to 0} \frac{\Phi(z+h) - \Phi(z)}{h} .$$

For this, the Cauchy-Riemann conditions are necessary,

$$\varphi_x(x, y) = \psi_y(x, y) , \quad \varphi_y(x, y) = -\psi_x(x, y) .$$

From the Schwarz theorem, it follows

$$\varphi_{xx}(x, y) = \psi_{yx}(x, y) = \psi_{xy}(x, y) = -\varphi_{yy}(x, y) ,$$

hence $\Delta\varphi(x, y) = \varphi_{xx}(x, y) + \varphi_{yy}(x, y) = 0$.
The same way we conclude that also ψ fulfills the Laplace equation.

Examples

Example

Consider the powers of z, $\Phi(z) = z^n$ with $n \in \mathbb{N}$.
Obviously, $\Phi'(z)$ exists, the known formula remains valid as in the real domain.
We have

$$z^n = (x + y\mathrm{i})^n = r^n(\cos(n\vartheta) + \mathrm{i}\sin(n\vartheta)) ,$$

where $z = r(\cos(\vartheta) + \mathrm{i}\sin(\vartheta))$.

Example

With, $\Phi(z) = \ln(z)$, and the derivative $\Phi'(z) = z^{-1}$, one obtains

$$\ln(z) = \ln(x + y\mathrm{i}) = \ln(r) + \mathrm{i}\vartheta ,$$

where we have to exclude the point $z = 0$, $x = y = 0$.

Visualization

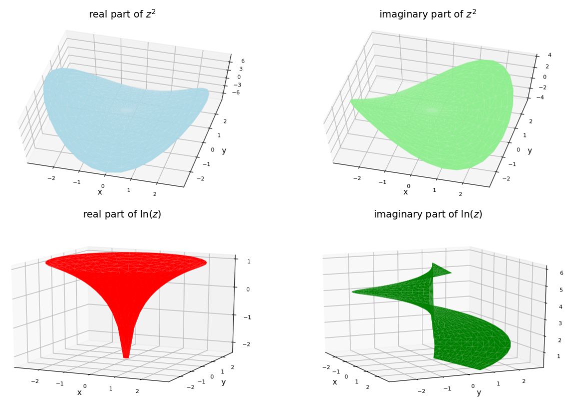

Fig. 25: Selected harmonic functions in two dimensions

Properties

Remark

From the examples one sees that harmonic functions do not have local extrema, and their mean curvature is zero.

It is not so obvious, but we will prove later, that the value of a harmonic function is equal to its mean value over any circle (sphere) or disk (ball) around that point.

Definition

By $B_r(x_c)$ and $S_r(x_c)$ we denote ball and sphere of radius r around a center located at x_c.

$$\begin{aligned} B_r(x_c) &= \{x : |x - x_c| < r\} , \\ S_r(x_c) &= \{x : |x - x_c| = r\} . \end{aligned}$$

Some facts

Notice that $B_r(x_c)$ is an open set, $S_r(x_c) = \partial B_r(x_c)$ is closed, and $cl\, B_r(x_c) = \overline{B_r(x_c)} = B_r(x_c) \cup S_r(x_c)$.

By mes(A) we denote the measure of a set A, e.g., in $\mathbb{R}^2$, the measure (area) of a ball $B_r(x_c)$ is πr^2, that of the sphere $S_r(x_c)$ is $2\pi r = \omega_2 r$.

The Lebesgue measure $mes(\cdot)$ is shift invariant, and it scales with r^d, where d is the dimension of the set (d=2 for the ball, d=1 for the sphere).

In the case of the 3d space, the volume $\text{mes}\,(B_r(x_c)) = \frac{4\pi}{3} r^3$, the surface area is $\text{mes}\,(S_r(x_c)) = 4\pi r^2 = \omega_3 r^2$.

Fundamental solutions

Definition (fundamental solution)

$$\Delta E(x) = \delta(x)\,,$$

where δ denotes Dirac's distribution.

Remark

We focus here on the Laplacian operator.
Analogous considerations can be made for other linear differential operators.

Remark

$$\int_{\mathbb{R}^n} \delta(x - x_0) f(x)\, dx = f(x_0) \quad \forall f \in C(\mathbb{R}^n)$$

Formulas

In the case of $\mathbb{R}^2$, the fundamental solution to the Laplace equation is

$$E(x) = \frac{1}{2\pi} \ln\left(\frac{1}{|x|}\right)$$

For all higher dimensions, it holds

$$E(x) = \frac{1}{(n-2)\omega_n} \frac{1}{|x|^{n-2}}$$

with

$$\omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$$

the $n-1$-dimensional measure of the unit sphere in $\mathbb{R}^n$, where

$$\Gamma(z) = \int_0^\infty x^{z-1} \exp(-x)\, dx \qquad \text{for } re(z) > 0\,.$$

Most needed cases

Remark (properties of Γ-function)

$$\Gamma(0.5) = \sqrt{\pi}\,, \qquad \Gamma(1) = \sqrt{1} \qquad \Gamma(1.5) = 0.5\sqrt{\pi}$$
$$\Gamma(z+1) = \Gamma(z) z$$
$$\Gamma(k) = (k-1)! \quad \forall k \in \mathbb{N}$$

Remark (fundamental solutions in 2d and 3d)

$$\text{2d:} \qquad E(x) = -\frac{1}{2\pi} \ln\left(\sqrt{x^2 + y^2}\right)$$
$$\text{3d:} \qquad E(x) = \frac{1}{4\pi}\left(\sqrt{x^2 + y^2 + z^2}\right)^{-1}$$

Check for the 2d case

With $r = \sqrt{x^2 + y^2}$, $\partial_x r = x / \sqrt{x^2 + y^2}$, we find

$$\begin{aligned}
\partial_x\left(-\frac{1}{2\pi}\ln\left(\sqrt{x^2+y^2}\right)\right) &= -\frac{1}{2\pi}\left(\sqrt{x^2+y^2}\right)^{-1}\frac{1}{2}\left(\sqrt{x^2+y^2}\right)^{-1} 2x \\
&= -\frac{1}{2\pi}\frac{1}{r^2}x \\
\partial_y\left(-\frac{1}{2\pi}\ln\left(\sqrt{x^2+y^2}\right)\right) &= -\frac{1}{2\pi}\frac{1}{r^2}y
\end{aligned}$$

$$\begin{aligned}
\Delta\left(-\frac{1}{2\pi}\ln\left(\sqrt{x^2+y^2}\right)\right) &= -\frac{1}{2\pi}\left(\frac{1}{r^2} + x\partial_x(r^{-2}) + y\partial_y(r^{-2})\right) \\
&= -\frac{1}{2\pi}\left(\frac{1}{r^2} - xr^{-4}x - yr^{-4}y\right) \\
&= 0
\end{aligned}$$

Check in polar coordinates

$$\begin{aligned}
\Delta E(x) &= \frac{1}{r}\partial_r(r\partial_r E(r,\vartheta)) + \frac{1}{r^2}\partial^2_{\vartheta^2} E(r,\vartheta) \\
&= -\frac{1}{2\pi}\frac{1}{r}\partial_r(r\partial_r \ln(r)) \\
&= -\frac{1}{2\pi}\frac{1}{r}\partial_r 1 \\
&= 0
\end{aligned}$$

Why the factor?

$$\nabla E(r,\vartheta) = \text{grad } E(r,\vartheta) = r^{-1}\vec{e}_r = \frac{1}{2\pi r}[x,y]$$

$$\begin{aligned}
\int_{B_1(0,0)} \Delta E(x,y)\, dxdy &= \int_{B_1(0,0)} \text{div grad} E(x,y)\, dxdy \\
&= \int_{S_1(0,0)} \text{grad } E(\vartheta)\vec{n}(\vartheta)\, d\sigma(\vartheta) \\
&= \int_{S_1(0,0)} \frac{1}{2\pi} 1^{-1}\vec{e}_r \cdot \vec{e}_r\, d\sigma(\vartheta) \\
&= 1
\end{aligned}$$

By the divergence theorem, the integral is the same for all domains containing the origin, i.e. the source of unit intensity, and zero for all domains, which do not have the origin in the closure.

Integral representation

Theorem (Poisson's formula)

For given g on $S_R(x_c) \subset \mathbb{R}^n$, the solution to the boundary value problem

$$\begin{aligned}
\Delta u(x) &= 0 \qquad \text{in } B_R(x_c)\,, \\
u(x) &= g(x) \qquad \text{on } S_R(x_c)
\end{aligned}$$

is given by

$$u(x) = \frac{1}{\omega_n R} \int_{S_R(x_c)} \frac{R^2 - |x - x_c|^2}{|x - y|^n} g(y)\, dy\,,$$

where ω_n is the surface measure of the $(n-1)$-dimensional unit sphere $S_1(0)$ in $\mathbb{R}^n$.

Implications

Corollary

Let $g : \mathbb{R} \to \mathbb{R}$ be 2π-periodic.
Then the solution to the 2d elliptic boundary-value problem for the Laplacian equation is given in polar coordinates by

$$u(r,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 + r^2 - 2rR\cos(\vartheta - \varphi)} g(\vartheta)\, d\vartheta$$

Remark

If we choose $x = x_c$, i.e. $r = 0$, we obtain the mean value theorem.

Illustration

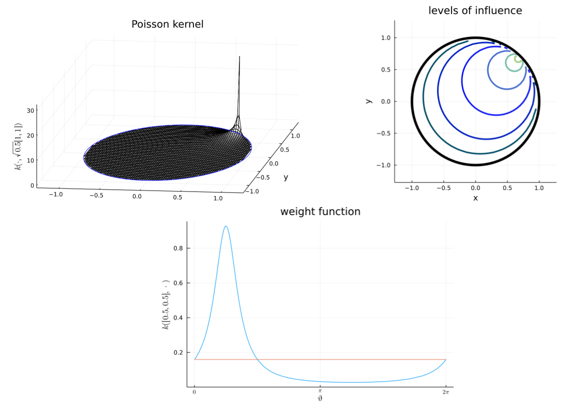

Fig. 26: Illustrations on Poisson's formula

Green's Representation Theorem

Theorem (Green)

Let u be harmonic on $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$, then

$$\forall x_0 \in \Omega \qquad u(x_0) = \int_{\partial\Omega} u(x)\frac{\partial E(x-x_0)}{\partial \vec{n}} - E(x-x_0)\frac{\partial u(x)}{\partial \vec{n}}\, dx\,.$$

Here E denotes the fundamental solution, $\vec{n}$ the outer unit normal to $\partial\Omega$

Remark (Green's function)

We may replace $E(x-x_0)$ by $G(x,x_0)$, where G is a functions with

$$\Delta G(x,x_0) = \delta(x-x_0)$$

and demand, additionally, e.g. $G(\cdot,x_0) = 0 \quad \forall x_0 \in \Omega$, or $\partial_{\vec{n}} G(\cdot,x_0) = 0 \quad \forall x_0 \in \Omega$, thus obtaining $u(x_0)$ in terms of boundary values of u or its normal derivative on the boundary.

Mean value theorem

Theorem (mean value theorem)

Let u be harmonic in $B_r(x_c)$ and continuous on $cl\, B = \overline{B}_r(x_c)$, then it holds

$$u(x_c) = \frac{1}{mes(B_r(x_c))}\int_{B_r(x_c)} u(x)\, dx\,,$$

$$u(x_c) = \frac{1}{mes(S_r(x_c))}\int_{S_r(x_c)} u(x)\, dx\,.$$

Remark

Notice that the measure is a different one for ball and sphere, $mes(S_r(x_c)) = \omega_n r^{n-1} = 2\pi^{\frac{n}{2}} r^{n-1}/\Gamma(\frac{n}{2})$, $mes(B_r(x_c)) = \pi^{\frac{n}{2}} r^n/\Gamma(1+\frac{n}{2})$, yet in both cases it is the integral of the constant function 1 over the considered set.
Both right-hand side have the interpretation of an integral mean value.

Maximum principle

From the previous properties, a maximum principle follows, similar as in the parabolic case. However, this time, by boundary we mean the topological boundary $\partial\Omega$ of the domain Ω.

Theorem (maximum principle)

Let u be harmonic in Ω and continuous in $\overline{\Omega}$. Then u attains its maximum and its minimum on the boundary $\partial\Omega$.
If, moreover, u is not constant, then u attains its maximum and minimum only on $\partial\Omega$.

Boundary value problems

Now, we study the Laplace equation

$$-\Delta u(x) = 0 \text{ in } \Omega$$

together with Dirichlet boundary conditions

$$u(x) = u_\Gamma(x) \text{ on } \partial\Omega\,,$$

where u_Γ is a prescribed function, defined on $\Gamma = \partial\Omega$.

Remark

The bounded domain $\Omega \subset \mathbb{R}^n$ and its boundary Γ need to meet some regularity conditions, if we expect a unique solution to the problem.

Remark

Alternatively, Neumann, Robin and mixed conditions may be imposed on Γ. Further, PDEs may be considered on unbounded domains, e.g. the exterior of a bounded Ω. Here we focus on bounded domains and the Dirichlet problem.

Existence and uniqueness

Theorem (well-posedness)

Under certain regularity conditions on the domain and the boundary function u_Γ, there exists a unique solution to the Dirichlet problem.

Remark

Classically, by solution, we mean a function u that is continuous on $\overline{\Omega}$ and has at least second order partial derivatives, so that the PDE is satisfied in each point of the domain, and the values are identical with those of the Dirichlet condition.

By a weak solution or variational solution, a more global approach is taken, e.g., the solution may be defined as minimizer of a certain (energy) functional.

Selected solutions

On the unit ball $B_1((0,0))^T$ in $\mathbb{R}^2$, for suitable boundary conditions, a solution can be obtained in the form of an infinite series.
Again, we use polar coordinates. The complex differentiable functions z^k can be written as

$$u(r,\vartheta) = r^k\cos(k\vartheta) \text{ and } u(r,\vartheta) = r^k\sin(k\vartheta)\,.$$

Combining them, using coefficients of the Fourier series of the boundary conditions, we arrive at

$$u(r,\vartheta) = \sum_{k=0}^{\infty} r^k\left(c_k\cos(k\vartheta) + d_k\sin(k\vartheta)\right)\,,$$

where

$$u_\Gamma(\vartheta) = u(1,\vartheta) = \sum_{k=0}^{\infty} c_k\cos(k\vartheta) + d_k\sin(k\vartheta)\,,$$

Quadratic BC

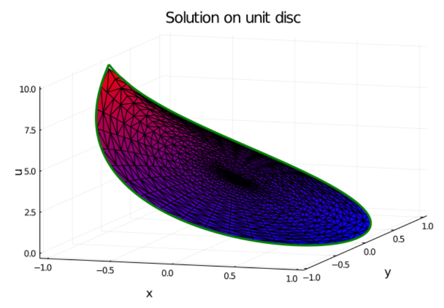

Fig. 27: Solution for $u_\Gamma(\vartheta) = \vartheta^2$

Poisson equation

The equation

$$-\Delta u(x) = f(x) \quad \text{ in } \Omega$$

for nontrivial f is called the Poisson equation.

It is studied as well with Dirichlet conditions, e.g. $u_{|\partial\Omega} = 0$.

Remark

Using linearity, solutions with nontrivial right-hand side and nontrivial boundary conditions may be composed of a solution to the Laplace equation, which satisfies the boundary conditions, and a second contribution, fulfilling the Poisson equation under zero boundary conditions.

Variational approach

If we multiply the PDE with a so-called test function (sufficiently smooth, vanishing on $\partial\Omega$), and integrate over Ω, we obtain equalities of the form

$$a(u, v) = b(v) \qquad \forall v\,,$$

which, ultimately, leads to the minimization problem

$$\frac{1}{2}a(u, u) - b(u) = \min\,.$$

Remark

The exact specification of the linear/bi-linear expressions and the domain of definition of the functional needs further explanation, too much for this introduction.