First Order Problems

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- vector fields and integral curves
- integral surfaces
- side conditions
- semi-linear PDEs
- quasi-linear PDEs
- initial-value problems
- characteristics and a nonlinear PDE
- difference methods
- consistency and stability

Homogeneous equations in two variables

We consider problems of the form

$$P(x,y)u_x(x,y) + Q(x,y)u_y(x,y) = 0\,,$$

where P and Q are given functions, defined in a suitable domain in the $x-y$-plane.
The wanted function u depends on the same variables as the given coefficients.

Homogeneous equations in three variables

The problems may be generalized to

$$P(x,y,z)u_x(x,y,z) + Q(x,y,z)u_y(x,y,z) + R(x,y,z)u_z(x,y,z) = 0\,,$$

in the $x-y-z$-space, and even too higher dimensions.

Remark (solution dependence)

Further, the coefficient functions may depend on the solution u as well.

Example (Burgers)

A well-known example is the inviscid Burgers equation

$$u_t(x,t) + u(x,t)u_x(x,t) = 0\,,$$

where the variables are renamed to x, t instead of x and y, the coefficient $Q(x,t)$ is constant equal to 1, while P is identical to the unknown solution $u(x,t)$.

Linearity

Definition (linear PDE)

If the unknown function u and all its derivatives occure only in a linear way with coefficients depending only on the independent variables, the equation is called a linear PDE.

Definition (semi-linear PDE)

If the derivatives of u occure only in a linear way with coefficients depending just on the independent variables, the equation is called a semi-linear PDE.

Definition (quasi-linear PDE)

If the highest order derivatives of the unknown function u enter the equation only in a linear way with coefficients depending solely on the independent variables and on lower order derivatives of u, the equation is called a quasi-linear PDE.

Inhomogeneous equations in two variables

If a source term (inhomogeneity) is given as right-hand side of the PDE, the equation becomes

$$P(x,y)u_x(x,y) + Q(x,y)u_y(x,y) = R(x,y)\,.$$

Remark (generalisations)

Again, the problem may be generalized to more variables (dimensions), and the coefficient functions may depend on the wanted function (solution-dependent coefficients, semi- or quasi-linear cases).

Linear to nonlinear

Example

$u_x + u_y = 1$ is linear
$u_x + u_y = u$ is linear
$u_x + u_y = u^2$ is semi-linear
$u_x + uu_y = 1$ is quasi-linear
$u_x + u_y^2 = 1$ is nonlinear

Uniqueness

In general, a PDE requires additional information to make its solution unique.
First order equations of the above form are mostly considered together with Cauchy conditions (initial conditions), defined on a curve (hyperplane) in the domain of independent variables.

Remark (well-posedness)

It turns out that not all manifolds are suitable for the definition of initial values.

Geometrical consideration

We start from the 2d homogeneous first order PDE

$$Pu_x + Qu_y = 0\,.$$

With $v = v(x,y) = (P,Q)$ and $\nabla u = (u_x, u_y)$, this can be understood as the orthogonality of the two vector fields v and ∇u (in the sense of their euclidean inner product vanishing on the whole of the domain of definition).

Isolines

Let the function f of two variables x and y be differentiable, and its gradient be denoted by $g = g(x) = \nabla f = (f_x, f_y)$.
In all points (x, y) of the domain, where $g(x, y)$ does not vanish, the set

$$\{(\xi, \nu) \,:\, f(\xi, \nu) = f(x, y)\}$$

is locally a curve orthogonal to $g(x, y)$.

Corollary

If f is such that ∇f is orthogonal to $v = (P, Q)$, then $f(x, y)$ solves the PDE, and also all functions h of f, i.e. $u(x, y) = (h \circ f)(x, y)$ solve the PDE $Pu_x + Qu_y = 0$.

Example

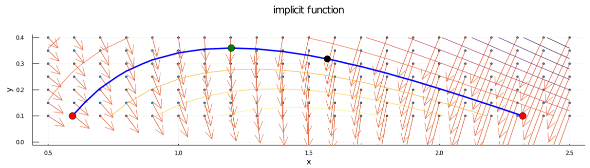

Fig. 2: Vectorfield and levels

Integral curves

Definition (trajectory)

A curve $\{(x(s), y(s))\}$ is called an integral curve or trajectory of a vectorfield v if

$$\frac{d}{ds}(x(s), y(s)) \parallel v(x(s), y(s)) \qquad \forall s$$

If we can find trajectories to $v = (P, Q)$ in implicit form $C(x, y) = const$, all functions $u(x, y) = h(C(x, y))$ will be solutions to $Pu_x + Qu_y = 0$.

Characteristic equation

Trajectories of the vectorfield $v = (P, Q)$ can be found in parametric form by solving the system of ordinary differential equations

$$\begin{aligned} \frac{dx}{ds} &= P\,, \\ \frac{dy}{ds} &= Q\,. \end{aligned}$$

Alternatively, one can consider

$$\frac{dx}{P} = \frac{dy}{Q}\,.$$

In this case, the parameter s of the trajectory is eliminated, but zeros of P and Q need some attention.

Remark

In fact, we can rewrite the symmetric form as

$$\frac{dy}{dx} = \frac{Q}{P} \quad \textit{or} \quad \frac{dx}{dy} = \frac{P}{Q}\,.$$

Example

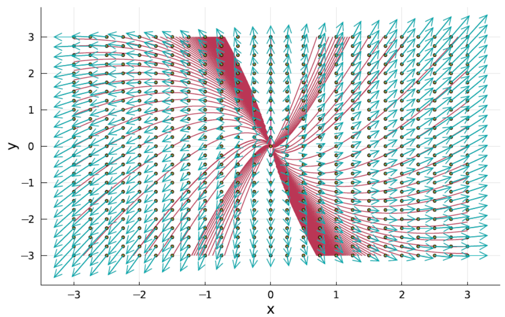

Fig. 3: Integral curves – 2d-case

Initial conditions

In order to find particular solutions to the system of ODEs or to the single ODE, initial conditions need to be specified.

To this end, on a curve that is exactly once intersected by each of the integral curves, values of the unknown are assigned. Along the trajectories (integral curves, characteristics), these values do not change.

Transport equation

Example (transport equation)

$$u_x + u_y = 0$$

vectorfield: $(1, 1)$
trajectories: $x(s) = s + x_0$, $y(s) = s + y_0$, or $y = x + c$
implicit form: $C(x, y) = y - x = c = const$

The solution to the transport equation is constant along characteristic lines, which travel at constant speed down the space variable (here x, if we think of y as time variable t).

Remark

If we assign values $u_0(x)$ on the straight line $y = 0$, the solution is completely determined, e.g. for $u_0 = \arctan(x)$ we obtain

$$u(x, y) = \arctan(x - y)\,.$$

Recipe

Remark

In order to determine the value of the solution u at a point (x, y), we follow the integral curve passing through this position until we hit the curve, where initial data is given. There we look up the value and assign it to $u(x, y)$.

Remark

Integral curves do not intersect or branch for the given type of problems, assuming mild regularity conditions. Hence, the initial curve should not be itself a characteristic (trajectory, integral curve) – we would never reach it.

Remark

If side conditions are prescribed on a characteristic, the data there have to be consistent with the ODE (here: constant), and on the rest of the domain, the solution is not uniquely specified.

Spatial case

If we add a third independent variable z and also a third component to the vectorfield v, we need to consider integral surfaces.
In fact, in 3d space, there are two independent directions orthogonal to a given (nontrivial) vector.

Definition

A surface S is called an integral surface of a vectorfield v if in each point of S the vector v is tangential to S.

If we find two independent integral surfaces to a given v, their intersection will be an integral curve of v.
If we succeed to define integral surfaces in implicit form $C(x, y, z) = const$, $D(x, y, z) = const$, the general solution to the PDE $v \cdot \nabla u = 0$ can be represented in the form $F(C(x, y, z), D(x, y, z)) = 0$ with a smooth function F.

Initial condition

Remark

With some luck, the equation $F(C(x,y,z), D(x,y,z)) = 0$ can be transformed into an explicit condition of the form $D(x,y,z) = \Phi(C(x,y,z))$.

Remark

The function F, respectively Φ, has to be identified from initial conditions, given on a surface that itself is not an integral surface.

The recipe given in the 2d case remains valid: from a given position, follow the trajectory to the initial surface, pick up the value there and transport it (unchanged) to said position.

Source terms

If there appears an inhomogeneity, i.e. a source term on the right-hand side, the value of the wanted function grows along the integral curve at the rate defined by the source.

It turns out that there is an important relation between an inhomogeneous problem in the plane and a certain homogeneous problem in space.
This way, the solution to
$$Pz_x + Qz_x = R$$
can be found by considering
$$Pu_x + Qu_x + Ru_z = 0\,.$$

Equivalence

Theorem

A surface $\mathcal{S}$ given by $z = z(x,y)$ is an integral surface of $v = (P,Q,R)$ iff
$$P(x,y,z)z_x + Q(x,y,z)z_y = R(x,y,z)$$
A surface defined implicitly by $u(x,y,z) = c$ is an integral surface of v iff
$$P(x,y,z)u_x + Q(x,y,z)u_y + R(x,y,z)u_z = 0$$

Remark

It is much easier to solve the second PDE – it is for an unknown function u in three variables x, y, z, but it is linear and homogeneous.
Once $u = u(x,y,z)$ is found, the wanted function $z = z(x,y)$ has to be found by solving $u(x,y,z) = const$ for z.

Finding integral curves

There are two options to determine integral curves

- parametric form: $x = x(t)$, $y = y(t)$, $z = z(t)$
  solve the ODE system
  $\frac{d}{dt}(x(t), y(t), z(t)) = v(x(t), y(t), z(t))$
  $= (P(x(t),y(t),z(t)), Q(x(t),y(t),z(t)), R(x(t),y(t),z(t)))$
- implicit form: $u_1(x,y,z) = c_1$ and $u_2(x,y,z) = c_2$
  to get rid of the parameter t, consider the symmetric formulation
  $$\frac{dx}{P(x,y,z)} = \frac{dy}{Q(x,y,z)} = \frac{dz}{R(x,y,z)}\,.$$
  By solving two of the equations, e.g. $\frac{dy}{dx} = \frac{Q}{P}$ and $\frac{dz}{dx} = \frac{R}{P}$ we obtain two equalities
  $$u_1(x,y,z) = c_1 \quad \text{and} \quad u_2(x,y,z) = c_2\,.$$
  This means: the characteristic (integral) curves are defined as intersections of two integral surfaces.

Independence

Theorem

If the integral curves of a given vectorfield $v = (P,Q,R)$ are given as intersection of surfaces $u_1 = c_1$, $u_2 = c_2$, then u_1 and u_2 are solutions to the PDE
$$P(x,y,z)u_x + Q(x,y,z)u_y + R(x,y,z)u_z = 0\,.$$
Conversely, if u_1 and u_2 are functionally independent solutions, then the intersection of their level sets gives locally all integral curves.

Furthermore, all solutions can be found in the form
$$u(x,y,z) = F(u_1(x,y,z), u_2(x,y,z))\,,$$
where $F : R^2 \to R$ is any smooth real-valued function in two variables.

Example

Solve the initial value problem
$$2xy\frac{\partial z}{\partial x} + (y^2+1)\frac{\partial z}{\partial y} = 4xy^3 \text{ for } x, y \in \mathbb{R}\,, \text{ and } z(x,0) = x+1 \text{ for } x \in \mathbb{R}$$
First, we find the integral curves of the 2d vectorfield by separating the variables x and y and integrating
$$\frac{dx}{2xy} = \frac{dy}{y^2+1} \quad \rightsquigarrow \quad \frac{dx}{x} = \frac{2y\,dy}{y^2+1} \quad \rightsquigarrow \quad \frac{x}{y^2+1} = C$$

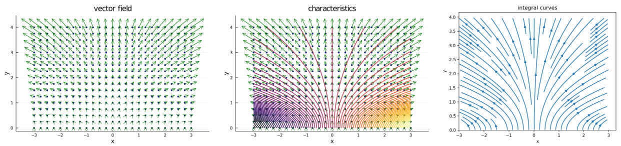

Fig. 4: Vectorfield, characteristics and stream

Example, continued

Next, we integrate z along the characteristics
$$\frac{dz}{4xy^3} = \frac{dy}{y^2+1} \quad \rightsquigarrow \quad dz = \frac{4xy^3\,dy}{y^2+1} = 4Cy^3\,dy \quad \rightsquigarrow \quad z(x,y) = \frac{xy^4}{y^2+1} + D\,.$$
Finally, one identifies the relation between the constants from the initial condition, arriving at
$$z(x,0) = x+1\,, \quad \text{which gives } D(C) = C+1\,.$$
Eventually, one obtains the wanted solution
$$z(x,y) = 1 + \frac{xy^4}{y^2+1} + \frac{x}{y^2+1} = \frac{x(y^4+1)+y^2+1}{y^2+1}\,.$$

Solution

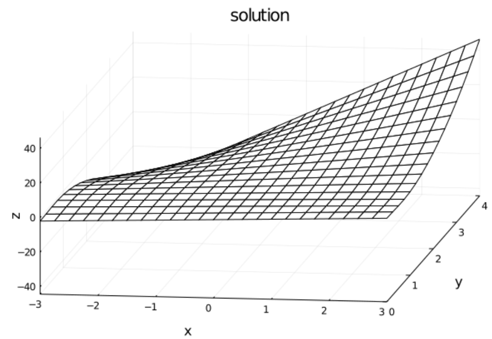

Fig. 5: Solution to the example problem