Second Order Linear PDEs

Classification and Canonical Forms

Second Order Problems

- general form and main symbol
- classification
- characteristic curves
- standard forms
- transformation to standard form

Linear second order PDE

Given a symmetric matrix $A \in \mathbb{R}^{n\times n}$ of coefficients in front of second order terms, a vector $b \in \mathbb{R}^n$ of coefficients for first order derivatives, and finally, a coefficient for the unknown function itself, we formulate the equation

$$Lu = f \quad \text{with} \quad Lu = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}u_{ij} + \sum_{i=1}^{n} b_i u_i + cu\,.$$

The operator, i.e. the coefficients A, b and c as well as the source term f may be functions of $x \in \mathbb{R}^n$.

The second order term $L_0 u = \sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n} A_{ij}u_{ij}$ is called the principal part of the operator. It decides about the main properties of the considered PDE.

Here we denote $u_i = \partial_{x_i}u(x) = \dfrac{\partial u}{\partial_{x_i}}$, $u_{ij} = u_{ji} = \partial_{x_i x_j}u(x) = \dfrac{\partial^2 u}{\partial_{x_i}\partial_{x_j}}$.

Examples

Example (Laplace)

Let $A = -I \in \mathbb{R}^{2\times 2}$, $b = 0 \in \mathbb{R}^2$, $c = f = 0$.

$$-\frac{\partial^2 u(x_1,x_2)}{\partial x_1^2} - \frac{\partial^2 u(x_1,x_2)}{\partial x_2^2} = 0\,.$$

Frequently, we use (x, y) instead of $x = (x_1, x_2)$ and write the equation in the form

$$-u_{xx} - u_{yy} = 0\,.$$

Remark (Δ)

This operator is defined also in higher dimensional cases, and denoted by Δ

$$\Delta u = \operatorname{div}\operatorname{grad} u = \sum_{i=1}^{n} u_{ii}\,,$$

The equation $-\Delta u = 0$ is called the Laplace equation.

Symbol

Different matrices lead to different PDEs, the spectrum of the matrix defines the type of the equation. Formally, we introduce the quadratic polynomial

$$L_0(x,\xi) = \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij}\xi_i\xi_j$$

as the principal symbol of a linear second order PDE.

Example

The principal symbol in the case of the Laplace equation $-\Delta u = 0$ is $-\|\xi\|^2$.

For the PDE $u_{x_1x_1} - u_{x_2x_2} = 0$, the polynomial $\xi_1^2 - \xi_2^2$ is obtained.

While the first polynomial has only the trivial root $\xi = 0$, the second one vanishes on two one-dimensional subspaces of $\mathbb{R}^2$. Each root ξ of $L_0(x,\xi)$ points – if nontrivial – in a direction that is characteristic for the equation under consideration.

Characteristic surfaces (curves in 2d)

Definition (characteristics)

A differentiable hypersurface in $\mathbb{R}^n$ such that its normal vector points in a characteristic direction of the principal symbol of a given PDE is called a characteristic surface (or characteristic curve if $n = 2$).

Obviously, the Laplacean operator Δ has no characteristic directions, so there are no characteristic surfaces or curves.

On the other hand, the one-dimensional wave equation $u_{xx} - u_{yy} = 0$ has the characteristic directions $(1, 1)$ and $(1, -1)$.
Consequently, the curves defined by $x + y = c$ and by $x - y = c$ are characteristic for this PDE.

Role of characteristics

Remark

In the case of first order PDEs, the principal symbol is $P\xi_1 + Q\xi_2$, and it has always nontrivial roots ξ. The characteristic directions are orthogonal to the vector (P, Q), thus the characteristic curves coincide with the trajectories (integral curves).

Likewise, in $\mathbb{R}^3$, the integral surfaces are characteristic surfaces.

Remember that

- characteristics are not suitable to pose initial data
- across a characteristic, solutions may have discontinuities (e.g. shocks)

In the second order case, characteristics may be used to introduce new variables, such that a PDE is transformed in a standard form.

Equation for characteristics

Assuming a characteristic surface (or curve) in the implicit form

$$w(x) = c$$

with a smooth scalar function $w : \mathbb{R}^n \to \mathbb{R}$, the normal vector has the direction $\nabla w(x)$ and is supposed to be nonzero.
The condition that it be characteristic leads to the equation

$$\sum_{i,j=1}^{n} A_{ij}(x)w_i(x)w_j(x) = 0 \quad \text{if } w(x) = 0\,,$$

which is a nonlinear first order PDE.

Again, we write briefly w_i for $\partial_{x_i}w(x)$.

Classification

Definition (types of PDEs)

A linear second order PDE with the principal symbol L_0 defined by the symmetric matrix $A(x)$ is

- elliptic if all eigenvalues of A are positive or all negative,
- parabolic if one eigenvalue of A is zero, all others are positive, or all negative,
- hyperbolic if all are nonzero, and exactly one has a different sign than the others.

Remark

If A is variable, the classification may differ from point to point.
We speak then of e.g. ellipticity in a given point x.
Otherwise, if the type does not depend on the location x, the operator (PDE) is elliptic, parabolic or hyperbolic in its whole domain $\Omega \subseteq \mathbb{R}^n$.

Special cases

Remark

A study of the spectrum is a considerable effort.

Ellipticity can be alternatively defined by the condition that the symbol has only the trivial root $\xi = 0$.

In the 2d-case, the following criterion may be used for the classification of an equation with principal part $\alpha u_{xx} + 2\beta u_{xy} + \gamma u_{yy}$, i.e.

$$A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix} .$$

A positive determinant of A means the PDE is elliptic, vanishing $\det(A)$ indicates parabolic, while a negative determinant corresponds to a hyperbolic equation.

Hint: The discriminant of the quadratic equation for the eigenvalues coincides with the determinant.

Canonical form

Definition (canonical form)

A linear second order PDE has a canonical form, if it does not contain any mixed derivatives.

In the special case of just two coordinates, $u = u(x, y)$, also a representation with only the mixed second order derivative $u_{xy}(x, y)$ and lower order terms is called canonical.

Example

The Poisson equation $u_{xx} + u_{yy} + u_{zz} = -10$ is an elliptic PDE in canonical form.
The heat equation $u_x = u_{yy} + u_{zz} + 10$ is a parabolic PDE in canonical form.
The wave equation $u_{xx} = u_{yy} + u_{zz}$ is a hyperbolic PDE in canonical form.
The PDE $u_{xy} = 3u_x - 5u_y + 3$ turns out to be hyperbolic, and its form is canonical, too.

Transformation

Assuming a constant matrix A of second order coefficients, using symmetry, we can always find an orthogonal matrix Q that diagonalizes A. The columns of Q are normalized eigenvectors of A, forming a basis of the linear space $\mathbb{R}^n$. It holds $A = Q\Lambda Q^T$, where Λ is a diagonal matrix comprising the eigenvalues of A.
Setting $x = Q\zeta$, $\zeta = Q^T x$, we can express $u(x) = u(x(\zeta)) = w(\zeta)$ in terms of the new variable ζ, and also all derivatives of u can be obtained in terms of derivatives of w with respect to ζ.

The PDE for $w = w(\zeta_1, \ldots \zeta_n)$ will have the principal part

$$\lambda_1 w_{11} + \lambda_2 w_{22} + \ldots + \lambda_n w_{nn}$$

with $\lambda_i = \lambda_{ii}$ the eigenvalues of A.

Invariance

Remark

A transformation of variables does not change the type of the PDE.

In fact, the spectrum is the same before and after the transformation.

Remark

In the special case of just two independent variables, even when coefficients are variable, $A = A(x, y)$, transformation to a canonical form may be handled analytically.

Transformations of coordinates will be then nonlinear, and tricky to construct. However, they also leave the type of PDE invariant.

The two-dimensional case

In two variables, x and y, the principal part of a linear PDE reads

$$\alpha(x, y)u_{xx} + 2\beta(x, y)u_{xy} + \gamma(x, y)u_{yy} .$$

We look for new coordinates $\zeta(x, y)$, $\eta(x, y)$ such that the equation will take a canonical form, i.e., either the mixed derivative or both pure derivatives have zero coefficients.
Locally, the transformation is described by the Jacobian matrix

$$J(x, y) = \begin{pmatrix} \zeta_x & \zeta_y \\ \eta_x & \eta_y \end{pmatrix} ,$$

which should be regular for the coordinates to be independent.

Transformed principal part

The matrix A defining the principal part transforms to $J^T A J$ plus first order terms, thus the determinant of the new matrix after transformation will have the same sign as before, $\det(J^T A J) = \det(J)^2 \det(A)$.

The PDE becomes

$$\begin{aligned} & (\alpha\zeta_x^2 + 2\beta\zeta_x\zeta_y + \gamma\zeta_y^2)w_{\zeta\zeta} \\ + \quad & (\alpha\zeta_x\eta_x + \beta(\zeta_y + \eta_x) + \gamma\zeta_y\eta_y)w_{\zeta\eta} \\ + \quad & (\alpha\eta_x^2 + 2\beta\eta_x\eta_y + \gamma\eta_y^2)w_{\eta\eta} + \cdots = g(\zeta, \eta) . \end{aligned}$$

The requirement for an implicitly defined curve $\nu(x, y) = c$ to be a characteristic of the PDE has the form

$$\alpha\nu_x^2 + 2\beta\nu_x\nu_y + \gamma\nu_y^2 = 0 .$$

It turns out to be useful to calculate characteristics in order to find suitable new coordinates.

Constructing the coordinate system

Assuming that $\nu_y \neq 0$, the condition becomes

$$\alpha\left(\frac{\nu_x}{\nu_y}\right)^2 + 2\beta\frac{\nu_x}{\nu_y} + \gamma = 0 ,$$

hence

$$\frac{\nu_x}{\nu_y} = \frac{-\beta \pm \sqrt{\beta^2 - \alpha\gamma}}{\alpha} .$$

We solve the ODE

$$\frac{dy}{dx} = -\mu_{1,2}(x, y) ,$$

where $\mu_{1,2}$ are roots of the quadratic equation $\alpha\mu^2 + 2\beta\mu + \gamma = 0$, and represent the solutions implicitly as $\varphi_{1,2}(x, y) = c$.

The curves $\zeta(x, y) = \varphi_1(x, y) = c$, $\eta(x, y) = \varphi_2(x, y) = d$, form families of characteristics.

Special cases

Remark

Only in the hyperbolic case one obtains that way two independent new coordinates.
In this case, by construction, the principal part reduces to mixed derivatives, only.

If the discriminant $\beta^2 - \alpha\gamma$ is zero, the PDE is parabolic, and only one family of characteristics can be found.
We use one of the original variables together with the only new one as transformed coordinate system, e.g. $\zeta = x$, $\eta = \varphi(x, y)$.

In the elliptic case, the discriminant is negative, and there are no real-valued solutions.
In this case, we proceed with the complex-valued pair of conjugate complex solutions, and we separate real and imaginary parts of the solution as new variables.

Example

Example (hyperbolic PDE)

$$xyu_{xx} - (x^2 + y^2)u_{xy} + xyu_{yy} = 0 .$$

We have to solve the quadratic equation

$$xy\mu^2 - (x^2 + y^2)\mu + xy = 0 ,$$

which leads to $\mu_1 = x/y$, $\mu_2 = y/x$, assuming $|x| \neq |y|$.
We find ζ and η by solving the two ODEs

$$y' = -x/y , \quad \text{respectively} \quad y' = -y/x .$$

Separating variables and integrating, one obtains $\zeta = x^2 + y^2$, $\eta = 2xy$.
The characteristics are a family of circles and a family of hyperbolas.

Example (continued)

In order to complete the example, all terms of the old PDE have to be expressed in the new coordinates.
We apply the chain rule to $w(\zeta, \eta) = u(x(\zeta, \eta), y(\zeta, \eta))$, arriving at

$$
\begin{aligned}
u_x &= w_\zeta \zeta_x + w_\eta \eta_x = w_\zeta 2x + w_\eta 2y \,, \\
u_y &= w_\zeta \zeta_y + w_\eta \eta_y = w_\zeta 2y + w_\eta 2x \,, \\
u_{xx} &= 4x^2 w_{\zeta\zeta} + 8xy w_{\zeta\eta} + 4y^2 w_{\eta\eta} + 2w_\zeta \,, \\
u_{xy} &= 2\eta(w_{\zeta\zeta} + w_{\eta\eta}) + 4\zeta w_{\zeta\eta} + 2w_\eta \,, \\
u_{yy} &= 4y^2 w_{\zeta\zeta} + 8xy w_{\zeta\eta} + 4x^2 w_{\eta\eta} + 2w_\zeta \,.
\end{aligned}
$$

Finally, the initial left-hand side of the PDE, $xyu_{xx} - (x^2 + y^2)u_{xy} + xyu_{yy}$, becomes

$$4(\eta^2 - \zeta^2) w_{\zeta\eta} + 2\eta w_\zeta - 2\zeta w_\eta = 0 \,,$$

which makes it canonical.