Int. J. Engng Sci. Vol. 31, No. 3, pp. 385–395, 1993
Printed in Great Britain.

0020-7225/93 $6.00 + 0.00

FREE VIBRATIONS OF FINITE-MEMORY MATERIAL BEAMS

K. FRISCHMUTH
Fachbereich Mathematik der Universität Rostock, 2500 Rostock, Germany

W. KOSIŃSKI and T. LEKSZYCKI
Instytut Podstawowych Problemów Techniki, Polska Akademia Nauk (IPPT-PAN), ul.Świętokrzyska 21, PL-00-049 Warszawa, Poland

(Communicated by G. A. MAUGIN)

Abstract—Starting from the set of classical axioms of simple materials with memory in the sense of Coleman and Mizel (*Archs Rat. Mech. Anal.* **23**, 87 (1966), Ref. [1]), a model of a material with finite memory is constructed. The main problem in this construction is the choice of an appropriate space of histories. Since each continuous constitutive functional defined on the space corresponds to some material with finite memory, the class of possible models of materials is rather broad. For the practical application the problem of free vibration for layered vibrating beams composed of an elastic or viscoelastic core and symmetric damping layers made of a material described by a finite memory constitutive law, is considered. The paper is concluded with a summary of numerical results obtained up to now.

1. INTRODUCTION

The theory of materials with memory has proved to be a very proper instrument for a *qualitative* analysis of physical processes, especially those appearing in mechanics and thermodynamics (cf. [1–11], among others). For *practical application* and *numerical treatment,* however, one usually prefers constitutive–laws based either on the internal state variable theory [12–14], that is easier but more restrictive, or on the theory of differential type [4]. The necessity of handling with infinite histories, however, is in conflict with numerical requirements. This fact well explains the present state of art in applications of the constitutive model of a material with memory. For the class of *materials with finite memory* actually considered both theoretical and computational problems can be successfully treated in the present framework.

Section 2 of this paper begins with fundamental concepts of a model of materials with finite memory. The main theoretical problem there is that of constructing a space of histories in a way convenient for further identification and application. Since each continuous constitutive functional defined on the constructed space corresponds to some material with finite memory, the class of possible models of materials is rather broad. This is one of main properties of the present model of material with memory.

For the practical application we consider in Section 3 the problem of free vibration for three-layer vibrating beams (cf. [16]). Those beams are now composed of an elastic or visocelastic core and symmetric damping layers made of a material described by a finite memory constitutive law. Equations of motion and a variational principle for the complex amplitude function are derived in Sections 3 and 4. Iterative methods for calculating eigenvalues and eigensolutions are discussed in Section 5. In Section 6 we consider some details of a mathematical problem arising in the second step of the algorithm proposed in Section 5. The paper is concluded with a summary of numerical results obtained up to now and an outlook on the further investigation of optimal distributions of damping material over the length of a beam which has to be done in a forthcoming paper [18].

2. THE THEORY OF MATERIALS WITH FINITE MEMORY

Each constitutive modelling is based on the idea that a reaction (a response) depends on an action (a stimulus), and moreover that dependence is continuous, or even smooth in a sense.

In the Coleman–Mizel theory of simple materials with memory the reaction is given as a value of a constitutive operator (functional) defined on the space of infinite histories of inputs (e.g. deformations). Hence the continuity of the operator corresponds to a continuity of that dependence.

In the literature a continuity requirement is used for two different purposes.

(a) Take a constitutive operator that expresses the above dependence and use the continuity postulate to construct a state space (e.g. the space of action histories) in a way to make the operator continuous.

(b) Construct a state space or a history space together with its topology as an *a priori* notion and use the continuity requirement as a postulate for the construction of constitutive operator.

It turns out that the approach (a) was chosen in [7], while it was (b) in [1–3,8] and in several others places. We choose here the latter.

According to [1], the construction of a history space should be compatible with the following three requirements:

(CM1) Histories can be statically prolongated and moreover static prolongations of equivalent histories are equivalent, too.

(CM2) All sections of histories (i.e. restrictions of histories) are histories, i.e. are in the constructed space.

(CM3) Constant histories are in the space.

In fact, cancelling or even weakening one of those requirements is not acceptable. However, in [1–3] another requirement (a postulate) was implicitly assumed, namely,

(CM0) The history space is a Banach space—exactly a Köthe—Toeplitz space $\mathscr{B} = V/\approx$ with a norm given by a function norm ν, i.e.

$$\|\varphi\| = \nu(|\varphi|), \quad \text{and} \quad \varphi_1 \approx \varphi_2 \Leftrightarrow \nu(|\varphi - \varphi|) = 0 \quad \text{and} \quad V := \{\varphi : \nu(|\varphi|) < \infty\},$$

where each φ is a measurable function with domain R^+, on which a Borel measure μ is given, and where the norm of each element φ of $\mathscr{B}$ is compatible with that measure, i.e.

$$\|\varphi\| = 0 \text{ iff } \varphi(s) = 0 \ \mu\text{-almost everywhere on } R^+ := [0, \infty).$$

In other words: it is assumed that the history space is built up from all functions which make the norm $\|\cdot\|$ (the function norm ν) finite. Of course, the axioms of the norm of $\mathscr{B}$ (i.e. the axioms of the function norm ν) are by no means physical ones (cf. [2,3]).

Let us notice that until now no assumption on the type of memory of the material has been assumed. The memory could be fading if one introduces the fading memory postulate in the form of the Coleman–Mizel relaxation property (cf. [1–3]).

However, in order to build a suitable space of histories for modelling a material with finite memory it is apparently natural to replace the *fading memory postulate* of the Coleman–Mizel theory of materials with memory [1–3] by an appropriate and stronger *finite memory postulate*. However, the unplesant consequence of the postulate (CM0) (cf. Theorem 2.2 in [3]) is the following result

THEOREM. A state space $(\mathscr{B}, \|\cdot\|)$ satisfying (CM0–CM3) together with the additional finite memory postulate

$$\exists \omega > 0 \forall \varphi_1, \quad \varphi_2 \in \mathscr{B} \quad \varphi_1 |_{[0,\omega]} = \varphi_2 |_{[0,\omega]} \Rightarrow \varphi_1 = \varphi_2 \tag{FM}$$

is isomorphic to that of an elastic material, here $\varphi |_{[0,\omega]}$ denotes the restriction of φ, primitively defined on the whole R^+, to an interval $[0, \omega]$.

PROOF. From (FM) it follows (cf. [1]) that the influence measure μ vanishes on $[\omega, \infty)$, i.e. $\mu([\omega, \infty)) = 0$. On the other hand (CM0–CM3) yield the implication $\{\mu((a, b)) = 0 \Rightarrow \mu((0, \infty)) = 0\}$ with arbitrary $0 < a < b < \infty$ cf. [3]. Hence $\mu((0, \infty)) = 0$, but that in turn implies that two histories are equivalent if their final values coincide, i.e. at $s = 0$. The constitutive (response) operator is defined on the space of histories and hence does not distinguish equivalent histories; consequently the response depends only on the final value of deformation. That behaviour, however, is characteristic for elastic materials only.

This unexpected result makes reasonable the conclusion that the origin of that restriction lies in the general framework of the Coleman–Mizel theory.

Now, we are cancelling (CM0) assuming the following postulate, instead.

(P) The history space is a Banach (a Köthe–Toeplitz space) $(\mathscr{B}, \|\cdot\|)$ with $\mathscr{B} = V/\approx$, and with a norm given by a function norm ν as before (i.e. $\|\varphi\| = \nu(|\varphi|)$ and $\varphi_1 \approx \varphi_2$ iff $\nu(|\varphi - \varphi|) = 0$) and

$$V := \{\varphi : \nu(|\varphi_{(\sigma)}|) < \infty, \text{ for each } \sigma \in (0, \infty)\}.$$

Here $\varphi_{(\sigma)}$ denotes the *σ-section* of a history φ and as usually is given by $\varphi_{(\sigma)}(s) := \varphi(s + \sigma)$, for $s \in R^+$ and some $\sigma \geq 0$.

Let us notice, that now the requirement (CM2) is met as a consequence of (P); hence we have only to consider (P), (CM1) and (CM3). In Ref. [15] it was proved that in this case the state space of a material satisfying (P), (CM1) and (CM3) together with (FM) can be represented by a space of histories defined on the finite interval $[0, \omega]$.

In fact, let $\mathscr{F}$ be a Banach function space, the elements of which are μ' measurable functions on $[0, \infty)$, where the measure μ' is the restriction of μ to $[0, \omega]$. If $\|\cdot\|'$ denotes the norm in $\mathscr{F}$, then the first property of the finite history space is given by the following requirement:

(F1) If $\varphi_1, \varphi_2 \in \mathscr{F}$ and $\varphi_1 - \varphi_2 = 0$, then $\varphi_1(0) = \varphi_2(0)$.

From this requirement follows that the measure μ' must have an atom at $s = 0$. To formulate the next property, let us notice that since now a function φ from $\mathscr{F}$ can be regarded as a history (or more precisely—an equivalent class of histories) of finite duration, the static prolongation (continuation) $T^\sigma\varphi$ of φ by the amount of σ is defined by

$$(T^\sigma\varphi)(s) = \begin{cases} \varphi(0) & \text{if } s \leq \min(\sigma, \omega) \\ \varphi(s - \sigma) & \text{if } \sigma < s \leq \omega. \end{cases} \tag{2.1}$$

Note that the formula for T^σ does not have any sense for $\sigma > \omega$. However, for any $\sigma \leq \omega$ this map should be well defined in $\mathscr{F}$, and should be continuous, as its counterpart in the case of the infinite memory was (cf. [8]). Hence the next requirement will be:

(F2) For any $\sigma \in [0, \omega]$ the map T^σ defined by (2.1) is continuous as a map from $\mathscr{F}$ into $\mathscr{F}$.

The image of φ under the map T^σ can be regarded as the result of a composition of an element from $\mathscr{F}$ with a constant function $\varphi(0)$ on $[0, \sigma]$. In order to perform the composition of elements from $\mathscr{F}$ with nonconstant functions, called processes, one introduces a class Π of action-valued functions defined on the closed intervals of the type $[0, d]$, $d \geq 0$. That class is introduced in the way, which makes possible to prolongate a finite "history" by a process to get a new finite history. The properties of the prolongation are introduced by an additional postulate (cf. Postulate (F3) in [15]) and are natural for the model of material with finite memory. Thanks to this the properties of the class Π are similar to that required by Noll in his framework of "a new mathematical theory of materials" [6].

Let us consider a particular case of a material with finite memory. the influence measure μ' has the properties:

—it possesses an atom at $s = 0$,

—it is absolutely continuous on $(0, \infty)$,

—it vanishes on $(0, \infty)$.

Hence a linear constitutive functional on that history space has the form;

$$\underset{\sim}{r}(\varphi) = K\varphi(0) + \int_0^\infty k(s)\varphi(s)\, \mathrm{d}s,$$

where the kernel $k(s)$ vanishes for $s > \omega$ and K is a constant.

3. EQUATION OF MOTION FOR LAYERED BEAMS

We consider a layered beam composed of a core and damping symmetric layers with thickness $h_i(x)$, cross area $A_i(x)$ and inertial moment $I_i(x)$, $i = 1$ for the core and $i = 2$ for the damping layers. The configuration is assumed to be symmetric, cf. Fig. 1.

Due to the symmetry† we may assume

$$\varepsilon(x, y, z) = -yu''(x), \tag{3.1}$$

which together with the constitutive laws

$$\sigma(t) = a_i\varepsilon(t) + b_i\int_0^{\Delta_i} K_i(s)\varepsilon(t-s)\,\mathrm{d}s, \tag{3.2}$$

where $i = 1$ corresponds to the core, $i = 2$ corresponds to the damping layers, yields for the bending moment the expression

$$m(x, t) = -\sum_{i=1}^{2}\left[a_i I_i(x)u''(x, t) + b_i I_i\int_0^{\Delta_i} K_i(s)u''(x, t-s)\,\mathrm{d}s\right]. \tag{3.3}$$

Here Δ_i with $i = 1, 2$, denotes, in general different, finite memory temporal constants of the materials and correspond to ω appearing in the previous section. Moreover, if $\Delta_1 = 0$ then the core is elastic.

Now we are looking for solutions of the equation of motion of the beam

$$-m''(x, t) + [A_1(x)\rho_1 + A_2(x)\rho_2]u''(x, t) = 0 \tag{3.4}$$

in the form

$$\begin{aligned} m(x, t) &= M(x)\mathrm{e}^{\lambda t} \\ u(x, t) &= U(x)\mathrm{e}^{\lambda t}. \end{aligned} \tag{3.5}$$

Here the ρ_i are the densities of the corresponding layers. Equation (3.3) yields the relation between M and U

$$M(x) = -\sum_{i=1}^{2} i_i(x)\left[a_i + b_i\int_0^{\Delta_i} K_i(s)\mathrm{e}^{-\lambda s}\,\mathrm{d}s\right]U''(x). \tag{3.6}$$

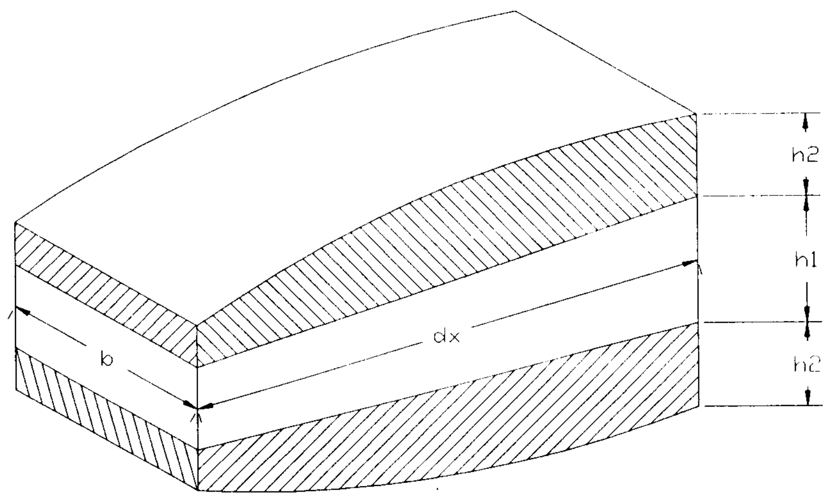

Fig. 1. An element of a layered beam.

†For nonsymmetric profiles and nonelastic materials this assumption may be violated, cf. [17].

Substituting (3.5) and (3.6) into the equation of motion (3.4) we obtain

$$\left[U''(x)\sum_{i=1}^{2} I_i(x)\left\{a_i + b_i\int_0^{\Delta_i} K_i(s)e^{-\lambda s}\,ds\right\}\right] + \lambda^2 U(x)\sum_{i=1}^{2} \rho_i A_i(x) = 0. \tag{3.7}$$

For brevity we denote

$$\mathcal{K}_\lambda(x) = \sum_{i=1}^{2} I_i(x)\left[a_i + b_i\int_0^{\Delta_i} K_i(s)e^{-\lambda}\,ds\right]$$

and

$$\mathcal{M}(x) = \sum_{i=1}^{2} \rho_i A_i(x). \tag{3.8}$$

Thus, we finally need to solve the ordinary differential equation

$$(\mathcal{K}_\lambda U'')'' + \lambda^2 \mathcal{M} U = 0. \tag{3.9}$$

As boundary conditions we assume here

$$\begin{aligned} U(0) = U(L) = 0 \\ U''(0) = U''(L) = 0, \end{aligned} \tag{3.10}$$

other cases can be treated in an analogous manner. Of course, the linear two point boundary-value problem (3.9), (3.10) always possesses the trivial solution $u \equiv 0$. Considering the free vibration problem, we want to find such values λ for which also a nontrivial solution in U exists. Doing this we must remember that the function $\mathcal{K}_\lambda$ depends on λ. Assuming that λ allows nontrivial solutions of (3.9), (3.10) we shall characterize them by a variational principle.

4. VARIATIONAL PRINCIPLE

In a standard way, (cf. [19,20]) we pass from the problem (3.9), (3.10) to a weak formulation: we first multiply (3.9) by a suitable test function V and then integrate it over the length of the beam

$$\forall V \int_0^L (\mathcal{K}_\lambda U'')''V\,dx + \lambda^2\int_0^L \mathcal{M}UV\,dx = 0. \tag{4.1}$$

Imposing on V the same homogeneous boundary conditions as on U in (3.10) and integrating twice by parts, we get

$$\forall V \int_0^L \mathcal{K}_\lambda U''V''\,dx + \lambda^2\int_0^L \mathcal{M}UV\,dx = 0. \tag{4.2}$$

The left-hand side of (4.2) is a symmetric sesquilinear form k_λ on $H_0^2(0, L)$ which depends on λ, hence we can equivalently rewrite (4.2) as

$$\forall V \in H_0^2(0, L) \qquad k_\lambda(U, V) = 0. \tag{4.3}$$

This is nothing else than the condition

$$\Pi_\lambda(U) = \frac{1}{2}k_\lambda(U, U) \text{ be stationary.} \tag{4.4}$$

Hence we have

THEOREM. The function $U \in H_0^2$ is a weak solution of the problem (3.9), (3.10) iff it is a stationary point of the functional

$$\Pi_\lambda(U) = \frac{1}{2}k_\lambda(U, U) = \frac{1}{2}\int_0^L \left(\mathcal{K}_\lambda U''^2 + \lambda^2\int_0^L \mathcal{M}U^2\right)dx.$$

REMARK. Π_λ is called the *complex potential energy* of the layered beam under consideration.

5. ITERATIVE METHOD

In the last section we have considered the case in which a complex number λ has been already known for the boundary-value problem (3.9), (3.10) that possesses nonvanishing solutions. Under this assumption the solutions are characterized by a variational principle. Let us now assume conversely that a nonvanishing amplitude function U is given which solves (3.9), (3.10)—at least approximately. Unfortunately, we cannot solve (3.9) for λ explicitly. On the other hand, it suffices to consider (4.3) for some appropriate V and to solve this scalar equation numerically for λ. If nothing better is at hand, the choice $V = U$ seems to be most natural, i.e. we find λ from the equation

$$\Pi_\lambda(U) = 0. \tag{5.1}$$

Now, the following iteration seems to be appealing: solve alternatively equation (5.1) for λ and (4.3) for a new U. However, the problem is that (4.3) possesses only for the exact value of λ nontrivial solutions. Hence, we must weaken the concept of solving the boundary-value problem in a suitable way for the iterative process. We do this in the following way. Equation (3.9) may be rewritten in the form

$$U = \frac{(\mathcal{K}_\lambda U'')''}{\lambda^2 \mathcal{M}}, \tag{5.2}$$

since $\lambda = 0$ is of no interest. Equation (5.2) has a fixed point form and may be used in two different ways for an iterative method. First, we can define a new approximation as the LHS of (5.2), substituting the only one of the RHS, and as an alternative we can substitute the old approximation on the LHS and solve (5.2) together with the boundary conditions (3.10) for a new approximation which occurs implicitly on the RHS. We choose the second possibility because it has the advantage that the corresponding iteration operator is bounded. Note also that for the real linear symmetric eigenvalue problem this method is known as an inverse power algorithm which converges to the fundamental eigenvalue. Hence, the iteration is defined by the following algorithm:

(1) Choose a suitable initial solution $U^{(0)}$ (kinematically admissible),
(2) Solve $\Pi_{\lambda^{(k)}}(U^{(k)}) = 0$ for $\lambda^{(k)}$,
(3) Solve $(\mathcal{K}_{\lambda^{(k)}} U^{(k+1)''})'' + \lambda^{(k)^2} \mathcal{M} U^{(k)} = 0$ for $U^{(k+1)}$ using the boundary conditions (3.10),
(4) Set $U^{(k+1)} = U^{(k+1)}/\|U^{(k+1)}\|$, $k = k + 1$ and go to (2.2).

This algorithm is extremely simple, it requires just the solution of (5.1) and the numerical integration problem in step (3). Nevertheless, there are some problems, too. So it will be pointed out in the next section that (5.1) possesses in general more than one solution, and the open problem is how to choose the proper one. We solve this problem in the next section.

6. SOLVING THE EQUATION FOR $\lambda^{(k)}$

The most crucial point of the algorithm described in Section 5 is the solution of the equation

$$\Pi_{\lambda^{(k)}}(U^{(k)}) = 0 \tag{6.1}$$

for $\lambda^{(k)}$. We have two basic algorithms at our disposal: Newton's method and simple iteration. For the simple iteration we just need to use the definition of Π (4.4) together with (4.2) and (4.3). So we rewrite (6.1) as

$$\lambda^2 = -\frac{\int_0^L \mathcal{K}_\lambda U''V''\,\mathrm{d}x}{\int_0^L \mathcal{M} UV\,\mathrm{d}x}. \tag{6.2}$$

This form is suitable for simple iteration after taking the square root. Taking one or another branch of the complex root we obtain two different iterations which in general converge even for same starting values to different solutions. We report on examples in Section 7.

In our implementation we use a globally convergent variant; for Newton's method, however, the most important is the choice of appropriate starting values. To this end we apply the implicit function theorem to (6.1). From this we obtain that

$$D\lambda_{|U^{(k-1)}}[U^{(k)} - U^{(k-1)}] = 2\frac{k_{\lambda^{(k-1)}}(U^{(k-1)}, U^{(k)} - U^{(k-1)})}{k'_{\lambda^{(k-1)}}(U^{(k-1)}, U^{(k-1)})}$$

where k' is defined analogously to k just with the kernel functions $(\mathrm{d}\mathcal{K}_\lambda/\mathrm{d}\lambda)$ and 2λ instead of $\mathcal{K}_\lambda$ and λ^2, respectively. Hence, in step (2) of the algorithm we can take as initial guess for $\lambda^{(k)}$ the following one:

$$\lambda^{(k)} = \lambda^{(k-1)} + 2\frac{k_{\lambda^{(k-1)}}(U^{(k-1)}, U^{(k)} - U^{(k-1)})}{k'_{\lambda^{(k-1)}}(U^{(k-1)}, U^{(k-1)})}.$$

7. NUMERICAL RESULTS

Two simple numerical examples are presented to illustrate the approach discussed in the preceding sections.

In the first example the influence of values of material constants, introduced in the finite memory model of material, on the first complex eigenvalue of a vibrating beam element is investigated, and some interesting observations are reported.

The aim of the calculations performed in the second example is to compare the first complex eigenvalues of vibrating beams with uniform and non-uniform damping layers of the same volume for different values of material damping.

7.1 Free vibration of a composite simply supported beam

Let us consider the free vibration of a simply supported composite beam and investigate its first complex eigenvalue. The beam has given length L, width b and consists of a solid elastic core of given constant thickness h_1 which is covered by two external layes of viscoelastic material of given thickness h_2 each of them.

In the numerical examples discussed in the present section we selected the particular form of the kernel K_i,

$$K_i(s) = \mathrm{e}^{-c_i s} - \mathrm{e}^{-c_i \Delta_i}. \tag{7.1}$$

For the sake of simplicity and to have easier mechanical interpretation of material parameters, let us introduce the following notation

$$a_i = \kappa_i E, \qquad b_i = -\kappa_i^2 E/\gamma_i, \qquad c_i = (\xi_i + \kappa_i)/\gamma_i \tag{7.2}$$

where E, κ_i, ξ_i and γ_i are constants. The parameters E, κ_i, ξ_i in some sense characterize elasticity, γ_i = material damping, whereas Δ_i is a measure of the finite memory.

For the elastic core material the value of the parameter γ_1 was taken to be almost equal to zero and two different viscoelastic materials of damping layers have been considered.

The first case of a material is similar to that one described by Voigt model but with finite memory ($\kappa_2 \gg \xi_2$). In Figs 2 and 3 the real and imaginary parts of the first eigenvalue versus finite memory parameter Δ_2 for different values of damping coefficient γ_2 are plotted.

Analogous drawings but for another material of damping layers—a material similar to one approximated by Zener model ($\kappa_2 = 10\xi_2$), are presented in Figs 4 and 5. In both cases the following values for densities and Young modulus have been selected, $\rho_1 = \rho_2 = 7800\,\mathrm{kg/m^3}$, $E = 2 \times 10^{11}\,\mathrm{N/m^2}$.

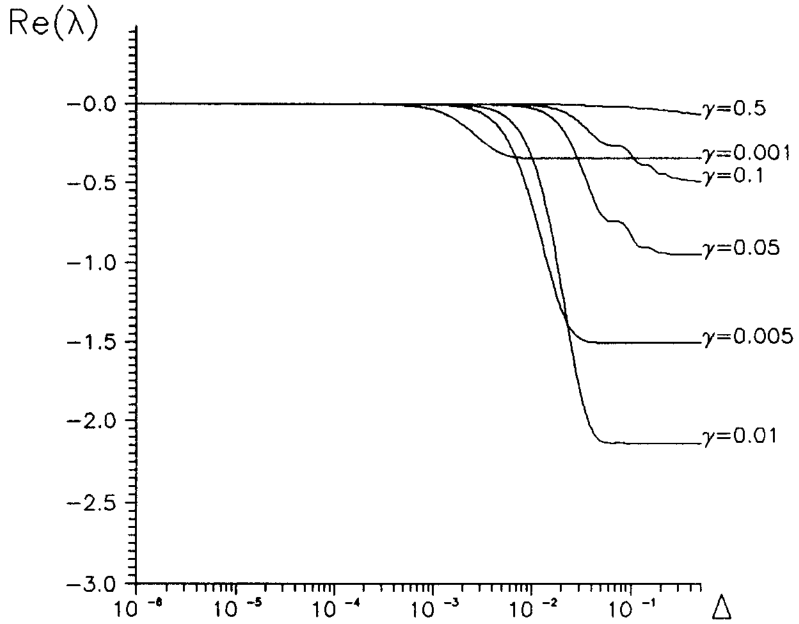

Fig. 2. Damping parameter of vibrations associated with fundamental eigenfrequency vs memory parameter for different values of material damping (material I).

As one can expect for very short material memory the material behaves similarly to elastic material. There is some transient domain for growing value of Δ where the results of analysis are very sensitive to the value of the finite memory material constant Δ. Moreover, many local extremes of real and imaginary parts of eigenvalues exist within this domain. For very big values of Δ materials considered behave as the viscoelastic material with infinite memory.

The numerical results suggest that an appropriate choice of the value for material memory plays an essential role in numerical calculations and results may vary considerably due to a strong dependence on the selected value of Δ.

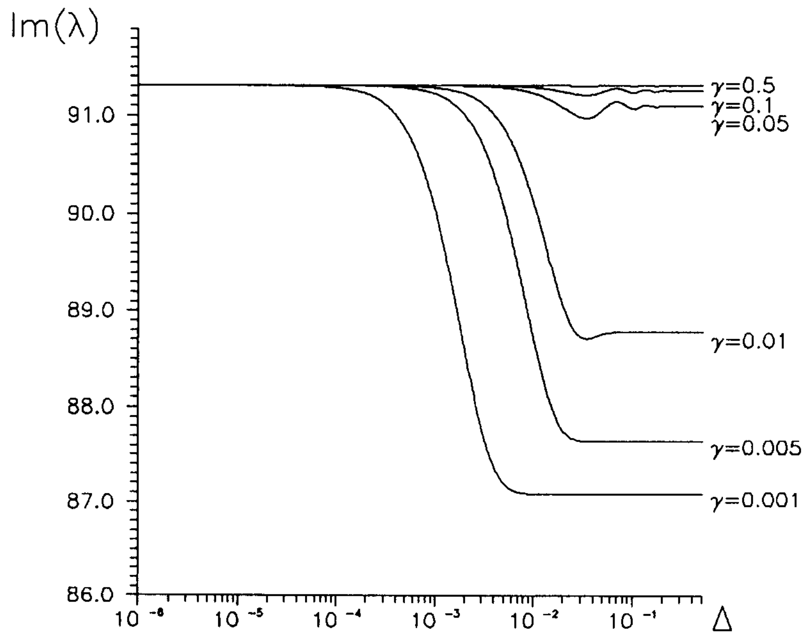

Fig. 3. Fundamental eigenfrequency vs memory parameter for different values of material damping (material I).

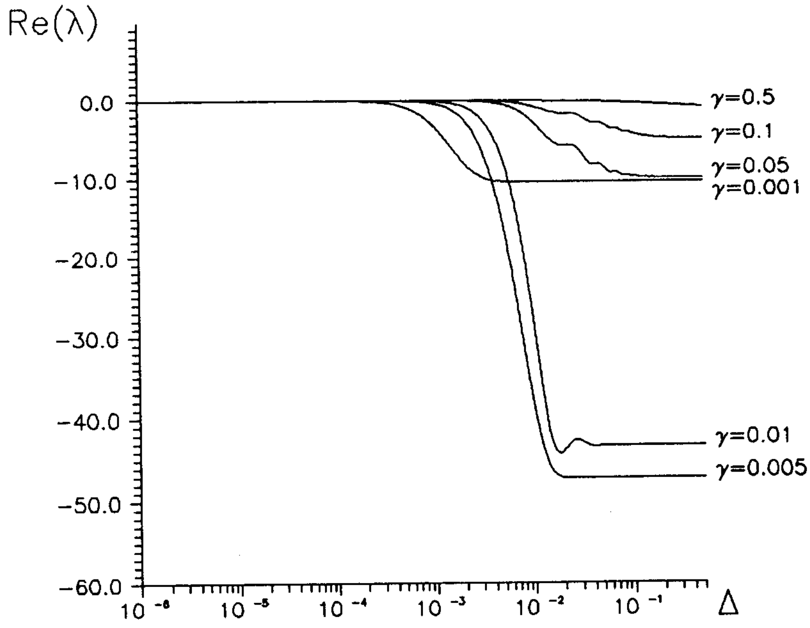

Fig. 4. Damping parameter of vibrations associated with fundamental eigenfrequency vs memory parameter for different values of material damping (material II).

7.2 *Vibration of a beam with variable thickness of damping layers*

Let us consider again a simply supported elastic beam covered with samping layers. The finite memory material of the layers satisfying assumption (7.1) is assumed and the influence of the value of the parameter γ_2 on the complex eigenvalue is investigated. For numerical calculations we assumed that $\kappa_2 = \xi_2$. The comparison of two different beams is done. The first one is again the beam with damping layers of constant thickness $h_2 = H$. The other one is a

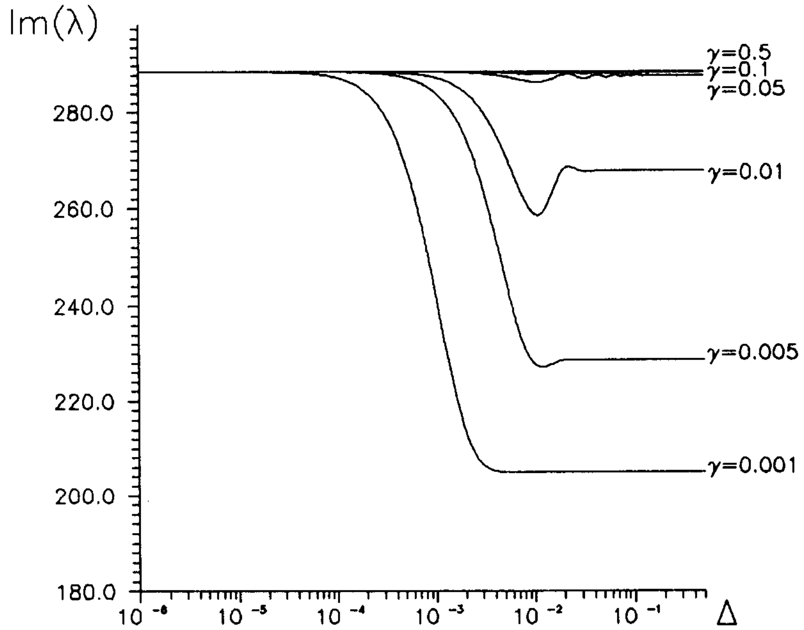

Fig. 5. Fundamental eigenfrequency vs memory parameter for different values of material damping (material II).

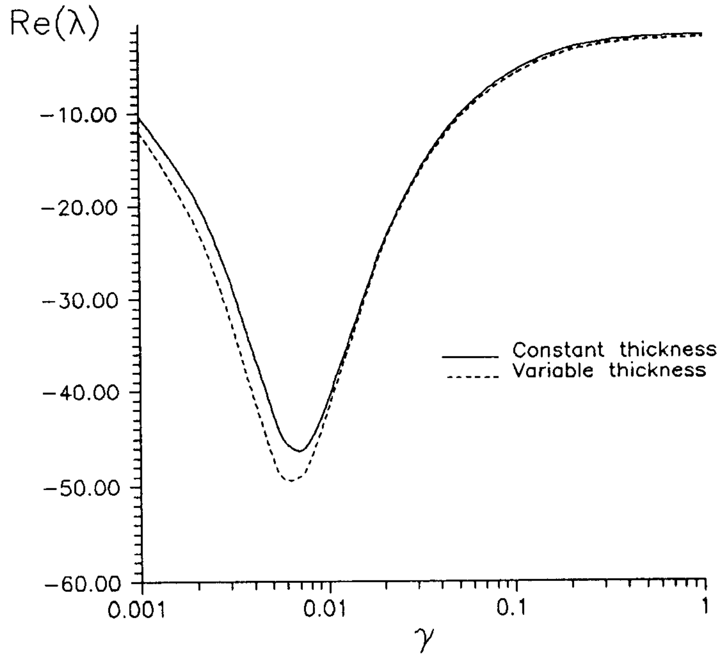

Fig. 6. Damping parameter of vibrations associated with fundamental eigenfrequency vs material damping for constant and variable thickness beams.

beam covered with layers of variable thickness defined as follows,

$$
\begin{array}{llll}
h_2 = H/2 & \text{for} \quad 0 \le x < L/6 & \text{or} & 5L/6 \le x \le L \\
h_2 = H & \quad L/6 \le x < L/3 & \text{or} & 2L/3 \le x < 5L/6 \\
h_2 = H/2 & \quad L/3 \le x < 2L/3 & &
\end{array}
$$

The changes of real and imaginary parts of eigenvalues of considered beams due to variation of damping coefficient γ_2 are presented in Figs 6 and 7, respectively.

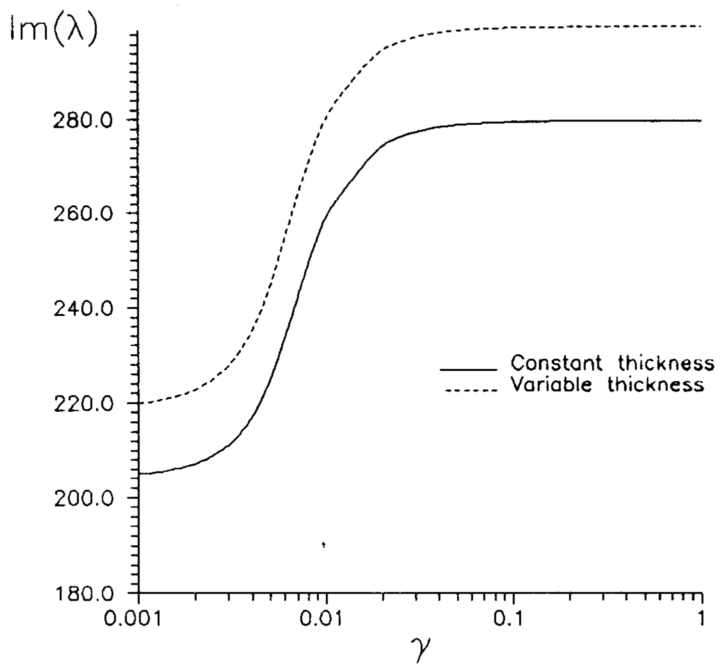

Fig. 7. Fundamental eigenfrequency vs material damping for constant and variable thickness beams.

As can be observed, there exists an optimal value of damping for which the damping decrement attains a minimum. It can be also seen that by appropriate distribution of material the dynamic properties of a beam can be improved. For a simple three-parameter model of viscoelastic material with infinite memory such an optimization problem has been formulated and solved for the case of forced vibration of one-dimensional elements of structures in [16].

Acknowledgements—Part of the work on this paper by the second author (WK) was done during his stay at Laboratoire de Modélisation en Mécanique, Université Pierre et Marie Curie, Paris VI, as a Professor Invité. The work of the third author (TL) was partly supported by the Polish Research Scientific Council under Grant no. 30939-91-01.

REFERENCES

[1] B. D. COLEMAN and V. J. MIZEL, *Archs Ration. Mech. Analysis* **23**, 87 (1966).
[2] B. D. COLEMAN and V. J. MIZEL, *Archs Ration. Mech. Analysis* **27**, 255 (1967).
[3] B. D. COLEMAN and V. J. MIZEL, *Archs Ration. Mech. Analysis* **29**, 18 (1968).
[4] C. TRUESDELL and W. NOLL, *The Non-Linear Field Theory of Mechanics*, Handbuch der Physik, Band III/3 (Edited by S. FLÜGGE). Springer, Berlin (1965).
[5] B. D. COLEMAN, *Archs Ration. Mech. Analysis* **17**, 1 (1964).
[6] W. NOLL, *Archs Ration. Mech. Analysis* **2**, 119 (1958).
[7] W. NOLL, *Archs Ration. Mech. Analysis* **48**, 1 (1972).
[8] K. FRISCHMUTH and W. KOSIŃSKI, *Archs Mech.* **34**, 515 (1982).
[9] W. KOSIŃSKI and K. C. VALANIS, *Archs Mech.* **35**, 541 (1983).
[10] B. D. COLEMAN and W. NOLL, *Archs Ration. Mech. Analysis* **6**, 355 (1960).
[11] A. E. GREEN and R. S. RIVLIN, *Archs Ration. Mech. Analysis* **1**, 1 (1957).
[12] W. KOSIŃSKI and W. WOJNO, *Archs Mech.* **25**, 709 (1973).
[13] K. C. VALANIS, *Irreversible Thermodynamics of Continuous Media. Internal Variable Theory.* Int. Centre Mech. Sci., Udine, Springer, Wien (July 1972).
[14] J. LUBLINER, *Acta Mech.* **17**, 109 (1973).
[15] K. FRISCHMUTH and W. KOSIŃSKI, *Archs Mech.* **41**, 543 (1989).
[16] T. LEKSZYCKI and N. OLHOFF, *J. Struct. Mech.* **9**, 363 (1981).
[17] K. FRISCHMUTH, *Rostock. Math. Kolloq.* **29**, 112 (1986).
[18] K. FRISCHMUTH, W. KOSIŃSKI and T. LEKSZYCKI, in preparation.
[19] T. LEKSZYCKI, *Mech. Struct. Mach.* **19**, 163 (1991).
[20] T. LEKSZYCKI and Z. MRÓZ, *SM Archs* **14**, 181 (1989).

(Received 3 April 1992; accepted 3 May 1992)