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On the triangulation of certain plane domains

1. Introduction

Numerical methods for solving two-dimensional systems of PDE's require in many cases a decomposition of the domain in triangles (cf. for instance [1 - 5]). There are various algorithms for the construction of triangulations and grid refinement, cf. [6, 7] . In the present paper we consider the triangulation of a special class of domains Ω which are called Braess domains due to their special meaning for the multigrid algorithm presented by Braess [8]. The properties of those domains are explicitly used by the algorithm to be described in Section 3 . Namely, we assume that

$$\partial\Omega = \bigcup_{j=1}^{m} S_j$$

with each S_j being a straight finite line satisfying

$$a_j x + b_j y = c_j \ , \ a_j, b_j \in \{0,1,-1\} \ , \ c_j \text{ integer.}$$

The initial point A_j of S_j is the endpoint of S_{j-1} (with index mod m); this condition defines the vertices A_j, $j = 1,\dots,m$, uniquely.
Now, our aim is to construct a decomposition of Ω in congruent triangles with edges of length 1, 1 and $\sqrt{2}$.

Usually FEM-codes require for the definition of the domain Ω the coordinates of the vertices A_j as data. However, for our special purpose we may assume that there are given the equivalent, but more convenient data

$$(K_l, \varkappa_l)\ ,\ l = 1,\ldots,k,$$

with

$$K_l = (x_l, y_l) \in \bar{\Omega}\ ,\ x_l, y_l \text{ integer},$$

$$K_l \neq K_{l'}\ \text{for } l \neq l',$$

$$\bigcup_{l=1}^{k} \{K_l\} = \bar{\Omega} \cap \mathbb{Z}^2, \quad \text{and}$$

$$\varkappa_l = \begin{cases} 0 & \text{for } K_l \in \Omega, \\ \text{anti-clockwise boundary point-counter for } Kl \in \partial\Omega, \end{cases}$$

where Ω is considered as an open set.

The enumeration of the boundary points starts at an arbitrary boundary point and encloses all integer boundary points. We denote the number of all integer boundary points by r. k is the number of grid points and n the number of triangles in the decomposition.

The main problem is now the nonconvexity of the domain Ω. We do not assume that $K_{l_1}, K_{l_2}, K_{l_3} \in \bar{\Omega}$ would imply $\text{conv}\{K_{l_1}, K_{l_2}, K_{l_3}\} \subset \bar{\Omega}$. Thus we need a test to decide whether a given triangle with edgelengths 1,1, $\sqrt{2}$ is contained in $\bar{\Omega}$ or not. In Sec. 2 we derive a criterion which is proper for a very economic testing procedure. This in turn is the basis for a decomposition algorithm, which is presented in Sec. 3 together with some examples demonstrating the power of the proposed criterion. The derivation of a 3-D analog of the test from Sec. 2 concludes the paper.

2. The criterion

Let us start with some simple but fundamental observations. As before Ω will be an open Braess domain.

Remark 1: A straight line with unit length and integer vertices is either entirely contained in $\bar{\Omega}$ or entirely contained in $\mathbb{R}^2 \setminus \Omega$.

Remark 2: A straight line with length $\sqrt{2}$ and integer vertices is

- either entirely contained in $\bar{\Omega}$ or
- entirely contained in $\mathbb{R}^2 \setminus \Omega$ or
- it is the union of exactly two parts of length $\frac{1}{2}\sqrt{2}$, one of them lying entirely in $\bar{\Omega}$, the remaining lying entirely in $\mathbb{R}^2 \setminus \Omega$.

Remark 1 and 2 imply in a straightforward way

Remark 3: A unit triangle, i.e. a triangle with integer vertices and edgelengths 1, 1, $\sqrt{2}$, is

- either entirely contained in $\bar{\Omega}$ or
- entirely contained in $\mathbb{R}^2 \setminus \Omega$ or
- it is the union of exactly two congruent triangles with edgelengths $\frac{1}{2}\sqrt{2}$, $\frac{1}{2}\sqrt{2}$, 1, one of them lying entirely in $\mathbb{R}^2 \setminus \Omega$ and the other lying entirely in $\bar{\Omega}$.

Remark 3 suggests using two testing points of the form

$$P_\alpha = \sum_{i=1}^{3} w_{\alpha_i} K_{1_i} \qquad i = 1,2,3 \ , \ \alpha = 1,2,$$

e.g. with

$$w_{\alpha_i} = \begin{cases} \frac{1}{3} \text{ near the right angle,} \\ \frac{1}{6} \text{ or } \frac{1}{2} \quad \text{else} \end{cases}$$

and

$$w_{1_i} + w_{2_i} = \frac{2}{3}, \quad i = 1, 2, 3,$$

and hence reducing the problem to that of deciding whether a given point belongs to Ω or not. This question, in turn, can be solved by calculating the index of $\partial\Omega$ with respect to the given point (cf. [12]). However, the drawback of this method is its cost. For each of the triangles to be tested two times an index must be calculated - which requires $O(r)$ operations. This motivates the following considerations which allow to avoid the use of testing points.

__Lemma 1:__ Consider the unit triangle $\Delta = \Delta K_{1_1}K_{1_2}K_{1_3}$. If two of the $\varkappa_{1_i}$'s vanish, then the triangle is contained in $\bar{\Omega}$.

__Proof:__ Assume $\Delta \not\subset \bar{\Omega}$, then according to Remark 3 two cases are possible. In the first case $\Delta \subset \mathbb{R}^2 \setminus \Omega$ and hence all its vertices belong to $\partial\Omega$. In the second one a subtriangle of Δ is contained in $\mathbb{R}^2 \setminus \Omega$ which has two common vertices with Δ. Thus if the assumption $\Delta \not\subset \bar{\Omega}$ is true, then at least two vertices of Δ belong to $\partial\Omega$ and hence at most one of the $\varkappa_{1_i}$'s may vanish. ■

However, the converse is not true, i.e. there may be triangles contained in $\bar{\Omega}$ with one or none vanishing $\varkappa_{1_i}$. We study such situations in the following three lemmas.

Lemma 2: Let $\varkappa_{l_1} = 0$ and $|\overline{K_{l_1} K_{l_2}}| = |\overline{K_{l_1} K_{l_3}}| = 1$. Then $\Delta = \Delta\, K_{l_1} K_{l_2} K_{l_3} \subset \bar{\Omega}$.

Proof: We consider again the two under $\Delta \not\subset \bar{\Omega}$ possible cases. In each of them there exists a subtriangle of Δ (in the first case Δ itself) with K_{l_1} as a vertex, which is contained in $\mathbb{R}^2 \setminus \Omega$. But this implies $\varkappa_{l_1} \neq 0$ – in contradiction to the assumption. ■

Lemma 3: Let $|\overline{K_{l_1} K_{l_2}}| = |\overline{K_{l_1} K_{l_3}}| = 1$ and $M = \frac{1}{2}K_{l_2} + \frac{1}{2}K_{l_3}$.
If $\overline{K_{l_1} M} \subset \partial\Omega$, then $\Delta \not\subset \bar{\Omega}$.
If $\overline{K_{l_1} M} \not\subset \partial\Omega$, then either $\Delta \subset \bar{\Omega}$ or $\Delta \subset \mathbb{R}^2 \setminus \Omega$.

Proof: It sufficies to observe that the third case of Remark 3 occurs iff $\overline{K_{l_1} M} \subset \partial\Omega$. ■

Lemma 4: Under the assumption of Lemma 3 we have $\overline{K_{l_1} M} \subset \partial\Omega$ iff $\exists\, l_4$ with $K_{l_4} = 2M - K_{l_1} \wedge \varkappa_{l_4} \neq 0 \wedge$ $\varkappa_{l_4} - \varkappa_{l_1} = \pm 1 \pmod r$.

Proof: a) Sufficiency. K_{l_1} and K_{l_4} are neighbouring boundary points, hence $\overline{K_{l_1} K_{l_4}} \subset \partial\Omega$, the more $\overline{K_{l_1} M} \subset \partial\Omega$.
b) Necessity. We have $|\overline{K_{l_1} M}| = \frac{1}{2}\sqrt{2}$. Therefore $\overline{K_{l_1} M} \subset \partial\Omega$ implies that $\overline{K_{l_1} M}$ is contained in a straight line belonging to $\partial\Omega$ with integer vertices and succesive numeration. ■

Now we are ready to formulate the main result.

__Theorem 1:__ Let $|\overline{K_{l_1}\,K_{l_2}}| = |\overline{K_{l_1}\,K_{l_3}}| = 1$ and $|\overline{K_{l_2}\,K_{l_3}}| = \sqrt{2}$. The triangle $\Delta = \Delta\, K_{l_1}K_{l_2}K_{l_3}$ is entirely contained in $\bar{\Omega}$ iff one of the following conditions holds :

(i) at least two of the κ_{l_i}, $i = 1,2,3,$ vanish;

(ii) κ_{l_1} vanishes;

(iii) $\kappa_{l_1} \neq 0$ $[\exists\; l_4$ with $K_{l_4} = K_{l_2} + K_{l_3} - K_{l_1}$ $\kappa_{l_4} \neq 0$ $\kappa_{l_4} - \kappa_{l_1} = \pm 1 \pmod r]$

$$[\kappa_{l_2}\kappa_{l_3} = 0 \quad \operatorname{sgn}(\kappa_{l_1}, \kappa_{l_2}, \kappa_{l_3}) \det\begin{pmatrix} 1 & x_{l_1} & y_{l_1} \\ 1 & x_{l_2} & y_{l_2} \\ 1 & x_{l_3} & y_{l_3} \end{pmatrix} = 1\;].$$

__Proof:__ Due to the lemmas it remaines to show that a triangle, which neither possesses two inner vertices, nor possesses K_{l_1} as an inner vertex, nor is cut up into two halfes by $\partial\Omega$, lies in $\bar{\Omega}$ exactly if there is an inner vertex or if the vertices (lying now all on $\partial\Omega$) form a positively oriented triangle with respect to the enumeration of the boundary. For the first case it suffices to observe that the existence of an inner vertex implies the existence of a non-empty common part of Δ and Ω - which excludes the possibility of $\Delta \subset \mathbb{R}^2 \setminus \Omega$. The second case is clear due to the anti-clockwise orientation of $\partial\Omega$. ■

3. The decomposition algorithm

Consider the following simple algorithm in step 1 of which we mean by "interesting":

- $\max |l_i - l_j| < B$,
- the edgelengths are $1, 1, \sqrt{2}$,
- each new generated triple has no inner points in common with the sum of the earlier triangles.

Here B is a suitable constant - depending on the numeration of gridpoints - which is easyly obtainable.

<u>Algorithm:</u> 1 generate all interesting triples of numbers of vertices l_1, l_2, l_3,

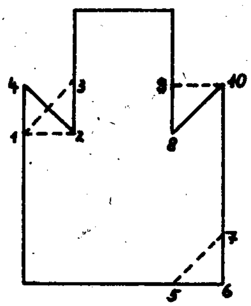
2 eliminate all triangles which are not entirely contained in Ω by using Theorem 1.

The cost of testing for interesting triangles may be reduced by prescribing the direction of hypotenuses, allowing exceptions only near $\partial\Omega$.

Step 2 of that algorithm is based on Theorem 1 from Sec. 2. Note that only if the case (iii) occurs the test will be rather expensive, in the worst case the position of two boundary points with given numbers $\neq l_1 \pm 1 \pmod r$ have to be checked for their position ($= K_{l_2} + K_{l_3} - K_{l_1}$?), the sign of a permutation and a determinant of order 3×3 are to be calculated.

Now, let us pass to some examples.

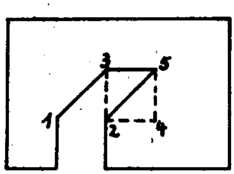
<u>Examples:</u>



$\Delta K_2K_3K_1$ is cut into 2 parts. K_2 and K_4 are neighbouring boundary points. The Δ is eliminated by (iii), first condition.

$\Delta K_6K_7K_5$ is accepted because it corresponds to the first case of Remark 3,

$\Delta K_9K_8K_{10}$ corresponds to the second case and is hence eliminated by (iii), second condition.



Here $\Delta K_2K_3K_1$ is eliminated by (iii), second condition, $\Delta K_2K_4K_3$ by (iii), first condition, while $\Delta K_4K_5K_6$ is accepted.

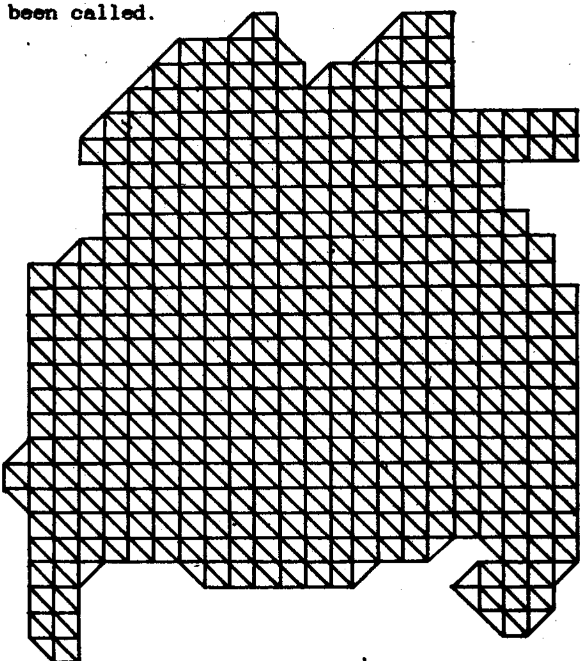
Our main interest was in the following example. Here Ω is an idealization of the Greifswald Bay containing 486 grid points and 869 triangles (cf [10]). The algorithm generated 287526 triples satisfying $\max |l_i - l_j| \leq B = 25$, rejecting 286648 candidates as not interesting in step 1 . From the remaining 878 were 764 accepted after testing (i),

28 accepted after testing (ii),

70 accepted after testing (iii), first condition,

9 rejected and 7 accepted after testing (iii), second conditon.

Hence, even if an index test would have been applied only in "doubtful" cases, i.e. if Lemma 1 and Lemma 2 do not give a decision, then 2*86 times a routine for index calculation would have been called.



Triangulation of the Greifswald Bay

4. Generalization to the 3-D case

Assume that the domain $\Omega \subset \mathbb{R}^3$ is bounded by a surface composed of pieces satisfying

$$c_\alpha^T x = b_\alpha \quad \alpha = 1, \ldots, m \qquad (*)$$

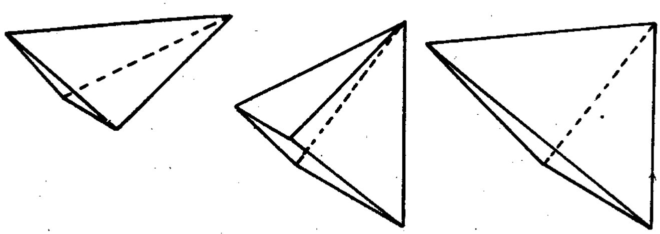
with

$$c_\alpha \in \left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}\pm1\\\pm1\\0\end{pmatrix}, \begin{pmatrix}\pm1\\0\\\pm1\end{pmatrix}, \begin{pmatrix}0\\\pm1\\\pm1\end{pmatrix}, \begin{pmatrix}\pm1\\\pm1\\\pm1\end{pmatrix} \right\}, \quad b_\alpha \in \mathbb{Z}.$$

Then the 3-D analog of the problem from Sec. 2 is that of deciding whether a unit tetrahedron with edgelengths 1, 1, 1, $\sqrt{2}$, $\sqrt{2}$, $\sqrt{2}$ and integer vertices is contained in Ω.

A tetrahedron of that type may be cut by a surface satisfying (*) in 6 different ways so that the discussion becomes a little more cumbersome. Possible cutting planes have unit normals from

$$\frac{1}{\sqrt{3}}\begin{pmatrix}\pm1\\\pm1\\\pm1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}\pm1\\\pm1\\0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}\pm1\\0\\\pm1\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}0\\\pm1\\\pm1\end{pmatrix}$$



Decompositon of unit tetrahedron by plane with normal vector $(1,1,-1)^T$.

A reinspection of Sec. 2 suggests that Lemma 1 and Lemma 2 should be easy to modify for the present case. However, the remainder of that section uses essentially the enumeration of the boundary, i.e. an explicitely given homeomorphism of $\partial\Omega$ with $\mathbb{R}/_{\mathrm{mod}\ r}$ or S^1. An analog of that would be an homeomorphism with S^2 in the spacial case. Unfortunately, it is rather unrealistic to assume such homeomorphism to be given or to be easy to calculate in cases of practical interest. Thus we are going to make weaker assumptions here, expecting in turn weaker results, i.e. at the outcome we shall have more expensive testing procedures. So we assume here that $\partial\Omega$ is connected and that an orientation of $\partial\Omega$ is given by assigning to each triangle $\Delta K_{l_1}K_{l_2}K_{l_3} \subset \partial\Omega$ an outer normal n_{l_1,l_2,l_3} of unit length.

Now we are going to prove two lemmas. This time we start with a 3-D counterpart of Lemma 2 .

<u>Lemma 2':</u> Let $|\overline{K_{l_1}\ K_{l_i}}| = 1$, $|\overline{K_{l_i}\ K_{l_j}}| = \sqrt{2}$, $i \neq j \in \{2,3,4\}$ and K_{l_1} be an inner point of Ω .
Then $\mathrm{conv}\{K_{l_i}, i = 1,\ldots,4\}$ is contained in $\bar{\Omega}$.

<u>Proof:</u> We observe that three of the faces of $\mathrm{conv}\{K_{l_i}\}$ are triangles of the type considered in Lemma 2 . Under the present assumptions the former ones are satisfied for these triangles, hence none of them may be cut by the boundary $\partial\Omega$, neither can not the tetrahedron $\mathrm{conv}\{K_{l_i}\}$. ■

<u>Remark 4:</u> A direct proof can by carried out analogously as in Lemma 2 considering the six possibilities of cutting up the tetrahedrons. K_{l_1} is always a vertex of each of the pieces.
Now it is easy to verify

Lemma 1': If no more than one of the points K_{1_i} is a boundary point, then $\operatorname{conv}\{K_{1_i}\} \subset \Omega$.

Proof: If again K_{1_1} is the distinguished vertex as in Lemma 2', then two cases are to be considered:

(a) K_{1_1} is an inner point – then Lemma 2' yields the thesis,

(b) K_{1_1} is not an inner point – then $K_{1_2}, K_{1_3}, K_{1_4}$ are inner points and hence the assumptions of Lemma 1 are fulfilled for the faces with edgelengths 1, 1, $\sqrt{2}$. Thus none of them is cut by the surface, consequently, the same is true for the tetrahedron. ■

In order to complete the discussion we need now a criterion deciding whether a unit tetrahedron with two or more boundary points among the vertices is cut by the surface or not, and another one deciding whether a tetrahedron lies inside or outside if the previous answer is no. For a better understanding imagine that we are considering e.g. the elasticity problem for RUBIC's cube twisted by $\frac{\pi}{4}$ per layer.

The first question may be solved introducing the six possible cutting triangles (with vertices from $\{K_{1_i},\ i = 1,\ldots,4,$ $K_{1_i} + K_{1_j} - K_{1_1},\ i \neq j = 2,3,4,\ K_{1_2} + K_{1_3} + K_{1_4} - 2K_{1_1}\})$ and checking, whether one of them belongs to the surface or not. We omit the details.

If the tetrahedron is not cut, and one of the K_{1_i} is an inner point, than again $\operatorname{conv}\{K_{1_i}\} \subset \Omega$. The remaining case is that of all K_{1_i} being boundary points. Now we are lacking the analog of the enumeration used in Theorem 1. Instead, we use the assumed to be given outer normal. We have the obvious

Theorem 2: Let $\mathrm{conv}\{K_{l_i}\}$ be a tetrahedron which is not cut by the surface $\partial\Omega$. Let $\Delta K_{l_1}K_{l_2}K_{l_3}$ be part of $\partial\Omega$. Then $\mathrm{conv}\{K_{l_i}\}$ is an inner tetrahedron iff

$$(n_{l_1,l_2,l_3}, \overrightarrow{K_{l_4}K_{l_1}}) > 0 .$$

In the remaining case we are forced to use rather a generalization of the testing point approach mentioned in Sec. 2. Namely we use then

Theorem 3: Let $\mathrm{conv}\{K_{l_i}\}$ be a tetrahedron which is not cut by the surface $\partial\Omega$, then it belongs to Ω iff

$$-\iint_{\partial\Omega} \frac{\partial}{\partial n_y} \frac{1}{|x-y|}\, d\omega(y) = 4\pi \qquad (**)$$

with $x = \frac{1}{4}\sum_{i=1}^{n} K_{l_i}$ and $n_y = n_{l_1',l_2',l_3'}$ for $y \in \Delta K_{l_1'}K_{l_2'}K_{l_3'}$.

Proof: The theorem is a direct consequence of the theorem on spacial angles, cf. [9, 11]. ■

Remark 5: For the criterion of a tetrahedron to be cut or not as well as for Theorems 2 and 3 it is essential to have an array of all boundary triangles together with the corresponding outer unit normals. To obtain those data we may use piecewise the modified plane decomposition procedure for each part of $\partial\Omega$. Doing this it is easy to incorporate the rule that

$$\vec{n} = \overrightarrow{K_{l_1}K_{l_2}} \times \overrightarrow{K_{l_1}K_{l_3}} \,/\, \|\overrightarrow{K_{l_1}K_{l_2}} \times \overrightarrow{K_{l_1}K_{l_3}}\| .$$

However, the 3-D case is essentially more complicated than the plane case since we are forced to avoid using global informations on the orientation. Fortunately, Theorem 3 must be used only in the case of very slender domains or domains with thin holes. Hence one may expect that the numerical approximation of (**) does not affect the overall costs of the decomposition algorithm to much.

Note that Theorems 2 and 3 are true for arbitrary tetrahedrons. In general a domain of the considered type cannot be decomposed using just unit tetrahedrons. Consequently, a 3-D-decomposition algorithm needs further tests. However, in most cases one can avoid using Theorem 3, especially, if further properties of Ω are used. So we have in many applications

$$\Omega = \{ - h(x,y) < z < 0 \ , \ (x,y) \in \Omega^{2D} \}$$

with a piecewise linear h. This makes a layer-by-layer decomposition possible which reduces the necessity of testing.

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