Arch. Mech., 38, 1-2, pp. 59-69, 1986

# Remarks on mathematical theory of materials

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THE AIM of the present paper is to clear up some of the originally introduced concepts by PERZYNA and KOSIŃSKI [6, 8] as well as to improve the definitions of a method of preparation in such a way, that the principle of determinism holds and the phenomena of plasticity and work-hardening fit into the theory.

Celem obecnej pracy jest wyjaśnienie pewnych oryginalnych koncepcji zawartych w pracach PERZYNY i KOSIŃSKIEGO [6, 8] oraz zaproponowanie nowych definicji dotyczących metody przygotowania. Wprowadzone nowe definicje zapewniają, że obowiązuje zasada determinizmu oraz w ramach proponowanej teorii można opisać takie zjawiska jak plastyczność oraz wzmocnienie materiału.

Целью настоящей работы является выяснение некоторых оригинальных концепций, содержавщихся в работах Пэжины и Косиньского [6, 8], а также предположение новых определений метода приготовления. Введенные новые определения обеспечивают факт обязывания принципа детерминизма, а также что, в рамках предложенной теории, можно описать такие явления, как пластичность и упрочнение материала.

## 1. Introduction

Any mathematical description, identification or modelling of physical object behaviours is based on some catalogue of observed phenomena. Modelling starts with a deterministic hypothesis which states, roughly speaking, "what depends on what", i.e., an input space G and an output space S are chosen. Having done this, the mentioned catalog is transformed into a table of input (i.e., G-valued) time-functions versus the corresponding output (i.e., S-valued) time-functions.

A central feature of inelastic systems is the non-uniqueness of the output. More specifically, to each input function $P$ defined on a bounded time-interval there corresponds, in general, a set of output functions $\mathrm{Z}_P$, such that for each $Z \in \mathrm{Z}_P$ the pair $(P, Z)$ belongs to the table of observations.

One way of associating a unique $Z$ with each $P$ consists in introducing such a parameter space K that $Z$ is determined by $P$ and an element of K through the response mapping $\mathscr{R}$: $(P, k) \mapsto Z$. Here the parameter $k \in \mathrm{K}$ is assumed to summarize the influence of the past inputs, that is before $P$ has started.

In the system theory the process of associating a S-valued function with each input is called either the parametrization of the space of input-output pairs (cf. ZADEH and DESOER [11]) or the state space realization (cf. WILLEMS [10]). It should be pointed out, however, that this process belongs to the modelling procedure forming one of its first steps.

Doing this step means to choose the state space approach. If the phenomena to be described are well understood and not too complex, the process of parametrization can be based on physical intuition only and the state space can be easily defined together with the map $\mathscr{R}$ as well as with a state transition function (called also an evolution function), where the latter governs the time evolution of states along input time-functions.

However, if the mechanisms governing the object behaviour are not entirely known, the problem of state space realization will be complex and, in addition to intuition, some more advanced mathematical methods will be necessary.

In continuum physics a number of theoretical models of deformable bodies are known. Using the concept of state, however, one can write down a master equation in terms of the map $\mathscr{R}$ for a sufficiently broad class of such models. In this way an order can be introduced in this "chaos" of constitutive models.

The idea of a state of a physical object is used formally or informally in almost all branches of physics. In continuum physics, however, the first use of the concept of state in a rigorous mathematical language was made by NOLL [7] in his New Theory of Simple Materials.

Stimulated by Noll, Perzyna and Kosiński published their alternative mathematical theory of materials in 1973. In their description the concept of a state arises as a consequence of the notion of a *method of preparation* and a configuration. Given in PERZYNA and KOSIŃSKI [8], the rules of interpretation of the first notion render their approach more adequate than the mathematically formal one of Noll.

However, some formal definitions following the concept of the method of preparation appearing in the original Perzyna–Kosiński theory as well as its thermodynamic generalization (cf. PERZYNA [9]) turns out to be too restrictive in describing plasticity and workhardening phenomena. On the other hand, they are too general to ensure the principle of determinism.

The aim of the present paper is to clear up some of the original concepts appearing in PERZYNA and KOSIŃSKI [6, 8] as well as to improve the definitions in such a way that the principle of determinism holds and the aforementioned phenomena fit into the theory.

## 2. The original definition

It is well known that for a nonelastic (dissipative) material system its response (i.e., output) depends on the way the system had been prepared before the input was applied. Furthermore, it is clear that each initial segment of the input time-function may be treated as a preparation of the system to the remaining segment of the input. These two observations will be of help in understanding what follows. Let us introduce a few definitions. If the sets G and S stand for the input space and the output space, respectively, then by the input time-function $P$ (or output time-function $Z$, respectively) we mean a G-valued function (or S-valued function) defined on $[0, \mathrm{dur}\, P]$, with $P \geqslant 0$. If G and S are equipped with some topologies, then only continuous functions will be considered. Note that in the case of the local theory of materials the sets G and S are subsets of finite-dimensional linear spaces, i.e. $\mathsf{G} \subset \mathrm{Sym}^+(\mathsf{T}, \mathsf{T}^*)$ and $\mathsf{S} \subset \mathrm{Sym}(\mathsf{T}^*, \mathsf{T})$, where T is a finite dimensional linear space.

Then any input time-function is called a *deformation process* and any output time-function is called a *stress process*. In what follows we often use these notions for a G-valued function or an S-valued function, respectively.

It should be noticed that the set of all deformation processes $\Pi$ (or the set of all stress processes Z, respectively) does not have the structure of a linear space because processes may differ in their durations(¹). However, one may define a composition operation for different processes $P_1$ and $P_2$ whenever $P_1(\operatorname{dur} P_1) = P_2(0)$; then the result called the continuation of $P_1$ with $P_2$ will be a new process $P_1 \divideontimes P_2$ naturally defined by

$$(P_1 \divideontimes P_2)(s) := \begin{cases} P_1(s), & 0 \leqslant s \leqslant \operatorname{dur} P_1, \\ P_2(s-\operatorname{dur} P_1), & \operatorname{dur} P_1 \leqslant s \leqslant \operatorname{dur} P_1 + \operatorname{dur} P_2 \equiv \operatorname{dur}(P_1 \divideontimes P_2). \end{cases}$$

Moreover, if $0 \leqslant t_1 \leqslant t_2 \leqslant \operatorname{dur} P$, then one can define a $[t_1, t_2]$-segment of a process $P$ as a new process $P_{[t_1,t_2]}$ as follows:

$$P_{[t_1,t_2]}(s) := P(s-t_2) \quad \text{for} \quad 0 \leqslant s \leqslant t_2 - t_1.$$

Coming back to the modelling, let us notice that the table of observed input processes versus the corresponding output processes may be treated as a subset R of the product $\Pi \times \mathrm{Z}$. Then, according to PERZYNA and KOSIŃSKI [8], the set K of all methods of preparation of a given material system (a material element or a specimen) is introduced together with the postulate that there exists a map $\mathscr{R}$ which realizes the relation R, i.e.

$$\exists \mathscr{R}: \Pi \times \mathrm{K} \to \mathrm{Z} \quad \forall (P, Z) \in \mathrm{R} \exists k_0 \in \mathrm{K} \quad \mathscr{R}(P, k_0) = Z \tag{2.1}$$

In the original paper mentioned above, the map $\mathscr{R}$ was defined on $(\Pi \times \mathrm{K} \times \mathrm{G})_{\mathrm{fit}} := \{(P, K, g) \in \Pi \times \mathrm{K} \times \mathrm{G} : P(0) = g\}$, which can be identified with $\Pi \times \mathrm{K}$. Consequently, the present approach is equivalent, but more economic. If $P = P_1 \divideontimes P_2$ with some $P_1, P_2 \in \Pi$, then $P_1$ is regarded as a preparation of $P_2$, and hence there should exist for each $k_0$ a new method of preparation $k_1$ such that

$$\mathscr{R}(P, k_0)_{[d_1, d_1+d_2]} = \mathscr{R}(P_2, k_1)$$

where $d_i := \operatorname{dur} P_i$, $i = 1, 2$. If the theory constructed is going to be deterministic, then the value $k_1$ should be determined by $k_0$ and $P_1$. So it is assumed that there exists a map $T$

$$T: K \times \Pi \to \mathrm{K}$$

called an *evolution function*, and such that

$$\mathscr{R}(P, k_0)_{[d_1, d_1+d_2]} = \mathscr{R}\big(P_2, T(k_0, P_1)\big). \tag{2.2}$$

The following uniqueness postulate is now introduced:

if $k_1 \neq k_2 \in \mathrm{K}$, then there exists at least one $P \in \Pi$ such that

$$\mathscr{R}(P, k_1) \neq \mathscr{R}(P, k_2). \tag{2.3}$$

Models which do not satisfy this postulate have been called by FRISCHMUTH ([2]): "prestructures". It is not difficult to prove that this postulate is weaker than

$$\forall P \in \Pi \ \mathscr{R}(P, \cdot): \mathrm{K} \to \mathrm{Z}_P \quad \text{is a bijection, where} \quad \mathrm{Z}_P := \{Z \in \mathscr{Z} : (P, Z) \in \mathrm{R}\}. \tag{2.4}$$

(¹) In KOSIŃSKI [4] the space of all processes is equipped with the structure of an infinite-dimensional differentiable manifold.

It should be noticed that in the most general case one assumes that (cf. paper(2) I) $\mathcal{R}(P, \cdot)$ is defined on $K_g$, with $g = P(0)$ and $K_g \subset K$. Then in the relation (2.1) the map $\mathcal{R}$ is defined on $\bigcup \{\Pi_g \times K_g : g \in G\}$, with $\Pi_g := \{P \in \Pi : P(0) = g\}$ and the state space $\Sigma$ is just the set $\bigcup \{\{g\} \times K_g : g \in G\}$.

In [8] the stronger condition (2.4) was introduced at once in the definition of the method of preparation space K. Furthermore, the evolution function was only defined, neither its existence nor uniqueness being discussed. In the next section we will show that, unfortunately, the postulate (2.4) leads to several complications in considering the evolution function.

On the other hand, Eq. (2.4) enables us to identify the states by simple measuring the actual values of the input and the output. Assuming the output space to be not greater than continuum (which usually is the case), in all special cases to which the theory from [8] applies the set K is not greater than continuum, too. Consequently, the classification of material structures given in [8] based on the cardinality of the method of preparation space K is not too reasonable. The introduced structure with internal state variables turns out to contain all other structures, while the class of non-trivial materials with memory (i.e., structures with a non-empty set A in Eq. (4.1) of [6]) is empty.

### 2.1. Main features of the original theory

In paper I the map $\mathcal{R}$ was used to define the instantaneous *response function* $\hat{S}$ (called in the system theory the read-out function, cf. WILLEMS [10]) as follows, if $s \in \operatorname{Dom} P$ and $(P, k) \in \Pi \times K$,

$$\mathcal{R}(P, K)(s) = \hat{S}(P(s), T(k, P_{[0,s]})). \tag{2.5}$$

In I the condition (2.5) was expressed in terms of $\hat{S}$ and the *evolution map* $\hat{e}$, since there $\Sigma \subset G \times K$. The present evolution function $T$ can be used to define $\hat{e}$ by the relation $\hat{e}((g, k), P) = (P(\operatorname{dur} P), T(k, P))$, for $(g, k, P) \in G \times K \times \Pi$. It is not proved in I whether $\hat{S}$ and $T$ are well defined by (2.5).

One of the weaknesses of the original Perzyna–Kosiński theory is the possibility of constructing a set K and a response map $\mathcal{R}$ for which no evolution function $T$ exists.

To show this let us take a model $\{K, \mathcal{R}\}$ for which an evolution function $T$ exists and which is non-trivial in the sense of

$$\exists_{K_1 \subset K} \ \exists_{k \in K_1} \ \exists_{P_1 \in \Pi} \ T(k, P_1) \notin K_1. \tag{2.6}$$

Note, that $K_1$ may be chosen as a singleton $\{k\}$. We assume that K, $\mathcal{R}$ and $T$ satisfy the conditions (2.2) and (2.4).

Now we are going to modify $\{K, \mathcal{R}\}$ defining a relation

$$R_1 := \{(P, \mathcal{R}(P, k) : P \in \Pi, k \in K_1\}$$

together with the set of methods of preparation $K_1$ and the response map $\mathcal{R}_1 := \mathcal{R}|_{\Pi \times K_1}$. The condition (2.4) is obviously satisfied and thus it remains to show the nonexistence of an evolution function $T_1$.

(2) In what follows the reference PERZYNA and KOSIŃSKI [8] is denoted by I.

To this end let us assume $T_1$ to exist and consider condition (2.2) for $P$, $k$ from (2.6). We obtain

$$\mathscr{R}_1(P,k)(\operatorname{dur}P) = \mathscr{R}_1(P^f_{(0)}, T_1(k,P))(0) = \mathscr{R}(P^f_{(0)}, T_1(k,P))(0)$$

with $T_1(k,P) \in \mathsf{K}_1$. Here $P^f_{(0)}$ denotes the process of zero duration with value $P^f := P(\operatorname{dur}P)$. Note that $P = P \mathbin{\divideontimes} P^f_{(0)}$. On the other hand, applying the condition (2.2) to the original model we arrive at

$$\mathscr{R}(P,k)(\operatorname{dur}P) = \mathscr{R}(P^f_{(0)}, T(k,P))(0) \quad \text{with } T(k,P) \notin \mathsf{K}_1.$$

Due to the bijectivity of $\mathscr{R}(P^f_{(0)}, \cdot)$, we may infer that

$$\mathscr{R}_1(P,k)(\operatorname{dur}P) \neq \mathscr{R}(P,k)(\operatorname{dur}P)$$

which contradicts the assumption, that $\mathscr{R}_1$ is just the restriction of $\mathscr{R}$ to the set $\Pi \times \mathsf{K}_1$.

REMARK. We used here the condition (2.2). This condition follows from (2.5) provided $T$ satisfies

$$T(k, P_1 \mathbin{\divideontimes} P_2) = T(T(k,P_1), P_2) \tag{2.7}$$

for all $k$, $P_1$, $P_2$ such that $P_1 \mathbin{\divideontimes} P_2$ exists.

In [2] the same fact was proved. In that paper the example of a semi-elastic material element (in the sense of [7]) was used.

The above case would be excluded if the postulate (2.4) were refused or weakened. There is, however, one more drawback connected with the plasticity phenomena. Let us prove the following

LEMMA. *If for a given response map* $\mathscr{R}$ *fulfilling* (2.4) *there exists a pair of mappings* $(T, \hat{S})$ satisfying (2.5), *then*

$$\underset{P\in\Pi}{\forall} \ \underset{Z_2\in Z_P}{\forall} \quad \text{if} \quad Z_{1[0,\varrho]} = Z_{2[0,\varrho]}$$

*for some* $\varrho \in \operatorname{Dom}P$ *then* $Z_1 = Z_2$.

Proof. Note first that if $Z = \mathscr{R}(P,k)$, then by Eq. (2.5) $Z_{[0,t]} = \mathscr{R}(P_{[0,t]}, k)$ for each $t \in \operatorname{Dom}P = \operatorname{Dom}Z$ and, moreover, by the relation (2.4) $k$ is the only element in $\mathsf{K}$ which satisfies the second equality. Let $Z_1 = \mathscr{R}(P,k_1)$ and $Z_2 = \mathscr{R}(P,k_2)$; then for any $t \in \operatorname{Dom}P$ we put $Z_{1[0,t]} = \mathscr{R}(P_{[0,t]}, k_1)$ and $Z_{2[0,t]} = \mathscr{R}(P_{[0,t]}, k_2)$.

From the above we conclude that if there exists $s \in \operatorname{Dom}P$ such that $Z_{1[0,s]} = Z_{2[0,s]}$, then $k_1 = k_2$ and consequently $Z_1 = Z_2$. □

It follows straightforward from Lemma that in the model satisfying the relations (2.4) and (2.5) the map $\hat{S}(g, \cdot): \mathsf{K} \to \mathsf{S}$ is invertible for any $g \in \mathsf{G}$. Since for the rate-type material element in the sense of Noll its intrinsic state is represented by a pair configuration — output $(g, S)$ with $g \in \mathsf{G}$ and $S \in \mathsf{S}$, we may conclude that each unique material structure in the sense of the paper I is of the rate-type.

It is not difficult, however, to observe that a plastic (or visco-plastic) material system with workhardening is not of the rate-type (cf. FRISCHMUTH [2]). Hence the original definition from the paper I is too restrictive and does not cover such a class of materials.

To conclude this section let us investigate implications following the condition (2.4) for the case of material systems with (fading) memory. As it was observed at the end of the previous section, the "classical" materials with memory in the sense of COLEMAN and MIZEL [1] fall into the class of material structures with internal state variables. This is in some sense reasonable but, nevertheless, it turns out that in modelling materials with memory, we can satisfy either the definition of the set of methods of preparation (2.1) or the definitions of the evolution function (2.2) and the response one (2.5).

To make it evident let us examine the consequence of the bijectivity of the map $\mathscr{R}(P, \cdot)$ for any $P$ for the case of a material with memory. It was not done in the paper I.

First of all, let us notice that "past history" and "method of preparation" are two different things, in general, since a whole class of "equivalent" in some sense (cf. FRISCHMUTH and KOSIŃSKI [3]) but different past histories correspond in general ([3]) to one method of preparation.

Let $\mathsf{K}^*$ denote the set of all past histories (i.e., functions defined on open interval $(0, \infty)$ with values in $\mathsf{G}$), and let $S^*$ be a response functional of a material with memory

$$S^*:\mathsf{G}\times\mathsf{K}^* \to \mathsf{S}.$$

In $\mathsf{K}^*$ there is an equivalence relation $\sim$ defined with the help of $S^*$ by

$$k_1^* \sim k_2^* \quad \text{iff } \forall P \in \Pi \; S^*\big(P(\operatorname{dur} P), k_1^* \ast P\big) = S^*\big(P(\operatorname{dur} P), k_2^* \ast P\big).$$

Denote by $k := [k^*]$ the equivalence class to which the history $k^*$ belongs. Then the evolution map $T$ is defined by

$$T(k, P) = [k^* \ast P] \tag{2.8}$$

where $k^* \in \mathsf{K}^*$

$$(k^* \ast P)(s) := \begin{cases} P(\operatorname{dur} P - s) & \text{for} \quad 0 < s \leqslant \operatorname{dur} P, \\ k^*(s - \operatorname{dur} P) & \text{for} \quad s > \operatorname{dur} P. \end{cases}$$

If $\beta:\mathsf{K}^* \to \mathsf{K}^*/_\sim =: \mathsf{K}$ is the canonical map, then the response function $\hat{S}:\mathsf{G}\times\mathsf{K} \to \mathsf{S}$ is simply defined by $\hat{S}(\cdot, \beta(\cdot)) := S^*(\cdot, \cdot)$. Let us notice that by applying Lemma to the map $\hat{S}$, we get the following implication for the map $S^*$:

$$\forall_{g\in\mathsf{G}} \;\; \forall_{k_1^*, k_2^* \in \mathsf{K}^*} \;\; \forall_{P\in\Pi_g} \quad \text{if} \quad S^*(g, k_1^*) = S^*(g, k_2^*) \quad \text{then} \tag{2.9}$$
$$S^*\big(P(\operatorname{dur} P), k_1^* \ast P\big) = S^*\big(P(\operatorname{dur} P), k_2^* \ast P\big)),$$

provided the response map $\mathscr{R}$ constructed from $\hat{S}$ and $T$ by (2.5) satisfies the relation (2.4).

Assuming the relation (2.9), consider $\mathsf{K}^*$ as the Lebesgue space $\mathsf{L}_{p,h}(0, \infty)$, $\mathrm{Sym}^+(\mathbf{R}^3))$ with $p > 1$ and the weight (the so-called influence function) $h:(0, \infty) \to \mathbf{R}^+$ such that $h(s)s^2 \to 0$, when $s \to \infty$. For further purposes assume that $S^*$ satisfies the strong principle of fading memory, which means that $S^*$ is once continuously differentiable. Then from the chain rule applying for the constant process $P(s) = \text{const} =: g \in \mathrm{Sym}^+(\mathbf{R}^3) \equiv \mathsf{G}$ and the equality $(2.9)_2$ we get,

$$\text{if} \quad S^*(g, k_1^*) = S^*(g, k_2^*) \quad \text{then} \quad \delta S^*(g, k_1^* | k_1^{*\prime}) = \delta S^*(g, k_2^* | k_2^{*\prime}), \tag{2.10}$$

where $\delta S^*$ is linear in the last argument and

$$-k_i^{*\prime}(s) := \frac{d}{ds} k_i^*(s) \quad \text{for a.e.} \quad s \in (0, \infty).$$

([3]) Past history is the same as a method of preparation only in the case of a material with permanent memory [3]. For such a material each of the two different histories will prepare the material differently for further response. The existence of such materials was recently shown by POHL and FRISCHMUTH [12].

Let us make the particular choice of a visco-elastic material function $S^*$ with a nonlinear instantaneous response, namely

$$(2.11)_1 \qquad S^*(g, k^*) = f(g) + H^*(g, k^*),$$

where

$$(2.11)_2 \qquad H^*(g, k^*) := \int_0^\infty Q(g, s)k^*(s)ds,$$

with $f \in C^1(\mathsf{G}, \mathsf{G})$ and $Q(g, \cdot)h^{-1}() \in \mathsf{L}_{q,h}$, where $q = p(p-1)^{-1}$.

Then the principle of fading memory holds and for the Fréchet derivative we get

$$\delta S^*(g, k_i^*|k_i^{*\prime}) = H^*(g, k_i^{*\prime}).$$

In view of the relation (2.10), the following implication is true, for any $g \in \mathsf{G}$:

$$(2.10)^* \qquad \text{if for some } k^* \in \mathsf{L}_{p,h} H^*(g, k^*) = 0 \text{ then } H^*(g, k^{*\prime}) = 0,$$

where $H^*(g, k^{*\prime})$ is defined only if $k^* \in \mathsf{D}$ with

$$\mathsf{D} := \{k^* \in \mathsf{L}_{p,h} : k^{*\prime} \in \mathsf{L}_{p,h}\} \subset \mathsf{L}_{p,h}.$$

The implication (2.10)* together with the definition (2.11) means that the following problem is stated, for any $g \in \mathsf{G}$: two linear maps $H^*(g, \cdot):\mathsf{L}_{p,h} \to \mathsf{S}$ and $M:\mathsf{D} \to \mathsf{L}_{p,h}$ with $Mk := k'$, $k \in \mathsf{D}$, such that

$$\ker H^*(g, \cdot)_D \subset \ker H^*(g, M(\cdot)).$$

Now the theorem on kernels tells us that there exists the linear map(4) $L:\mathsf{S} \to \mathsf{S}$ on the output space $\mathsf{S}$ which fulfills the following equality:

$$LH^*(g, k^*) = H^*(g, k^{*\prime}) \quad \text{for any} \quad k^* \in \mathsf{D}.$$

From the representation (2.11) we conclude that for any (test) function $k^* \in \overset{0}{C}_\infty(0, \infty)$

$$\int_0^\infty Q(g, s)k^{*\prime}(s)ds = \int_0^\infty LQ(g, s)k^*(s)ds,$$

which means that the distribution derivative of $Q(g, s)$ satisfies the equality

$$Q'(g, \cdot) = -LQ(g, \cdot).$$

In the standard way one concludes that $Q(g, \cdot)$ has the form

$$(2.12) \qquad Q(g, s) = Q_0(g)\exp(-Ls) \quad \text{for} \quad s \in (0, \infty), \text{ a .e.}$$

with $Q_0(g) \in \mathsf{S}$. This result was obtained originally by FRISCHMUTH [2] but by using less elegant arguments and stronger assumptions.

Looking at the last result one notices that the common part of the mathematical theory of Perzyna and Kosiński and the classical theory of materials with fading memory contains at least the class of Boltzmann materials. Moreover, restricting our attention to materials with linear dependence on the past history, one concludes that this common part is only composed of Boltzmann materials, however, with a general nonlinear instantenous response (i.e., with a nonlinear dependence on $g$). Since $\mathsf{S}$ is finite dimensional, we may try to de-

(4) Note that in general $L$ may be different for different $g \in \mathsf{G}$.

scribe the class of materials with memory governed by the relations $(2.11)_{1,2}$ in the form (2.12) by introducing a finite dimensional preparation method space, instead of the space of equivalent histories, and choosing an appriopriate evolution equation for internal state variables (i.e., for elements of the preparation method space).

## 3. The new definition of the set of methods of preparation

The contents of the previous sections form a proper basis to improve the original version of the Perzyna–Kosiński approach. Keeping two main ideas of that approach, namely, the concept of the method of preparation incorporated in the language of input-output processes, we are suggesting a new definition of the set $\mathsf{K}$ (cf. (2.1)).

To be more precise, let us list the properties of the relation $\mathsf{R} \subset \Pi \times \mathsf{Z}$ which is a mathematical idealization of the table of observed experiments. They are:

(i) if $(P, Z) \in \mathsf{R}$ then $\operatorname{dur} P = \operatorname{dur} Z$;

(ii) if $(P, Z) \in \mathsf{R}$ then for any $0 \leqslant t_1 \leqslant t_2 \leqslant \operatorname{dur} P$

$$(P_{[t_1, t_2]}, Z_{[t_1, t_2]}) \in \mathsf{R};$$

(iii) if $(P, Z) \in \mathsf{R}$ and $P \divideontimes P_1 \in \operatorname{Dom} \mathsf{R}$ then there exists a $Z_1 \in \mathsf{R}(P_1)$ such that

$$(P \divideontimes P_1, Z \divideontimes Z_1) \in \mathsf{R}.$$

Here we do not assume that Dom R is the whole $\Pi$ since for some models it may not be the case. Note that in general R is the so-called multi-function, i.e., $\mathsf{R}: \operatorname{Dom} \mathsf{R} \to 2^{\mathsf{Z}}$.

Comparing the conditions (i) — (iii) with that of the paper I, we can see the difference. In I the weaker form of (ii) was assumed; moreover the condition (iii) was not introduced in the original formulation of the theory. However, the condition (iii) was included in the revised version of the theory formulated by KOSIŃSKI [5]. Without the last condition it was possible (cf. FRISCHMUTH [2]) to construct an example of a system model without an evolution function.

Now let us pass to the realization of the set of methods of preparation. If we assume that there exists a nonempty set $\mathsf{K}$ and a map $\mathscr{R}: \Pi \times \mathsf{K} \to \mathsf{Z}$ such that

a) $\mathscr{R}(P, \mathsf{K}) \supset \mathsf{Z}_p := R(P)$, for any $P \in \operatorname{Dom} \mathsf{R} \subset \Pi$;

b) for any $k \in \mathsf{K}$ and $P$, $P_1 \in$ if $P_1 = P_{[0, \operatorname{dur} P_1]}$ then

$$\mathscr{R}(P_1, k) = \mathscr{R}(P, k)_{[0, \operatorname{dur} P]},$$

c) for any $(P, k) \in \Pi \times \mathsf{K}$ there exists exactly one $k_p \in \mathsf{K}$ such that for any $\bar{P} \in \Pi_g$, with $g = P(\operatorname{dur} P)$,

$$\mathscr{R}(\bar{P}, k_p) = \mathscr{R}(P \divideontimes \bar{P}, k)_{[\operatorname{dur} P, \operatorname{dur} P * \bar{P}]},$$

then $\mathsf{K}$ is called the *method of preparation space*, while $\mathscr{R}$ is called the *constitutive map.*

Comparing the present definition of the constitutive map with that of the paper I, we notice that no assumption concerning the bijectivity of $\mathscr{R}(P, \cdot)$ for any $P \in \Pi$ is assumed.

In the original formulation the condition

$$P \in \Pi: \mathscr{R}(P, k_1) = \mathscr{R}(P, k_2)$$