# ON THE THEORY AND APPLICATIONS OF A MATERIAL WITH FINITE MEMORY

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## INTRODUCTION

The theory of materials with memory has proved to be a very proper instrument for a qualitative analysis of physical processes, especially those appearing in mechanics and thermodynamics (cf.[1-9,11-12], among others). For practical applications and numerical treatment, however, one usually prefers constitutive laws based on the easier but more restrictive internal state variable theory [10,13,14]. The necessity of handling with infinite histories, which is in conflict with numerical requirements, well explains the present state of art in applications of the constitutive model of a material with memory. However, for the actually considered class of materials with finite memory both theoretical and computational problems can be successfully treated.
The main theoretical problem here is that of constructing a space of histories in a way convenient for futher identification and application. First of all each continuous constitutive functional defined on the constructed history space should correspond to some material with finite memory. An apparently natural approach in getting this is the substitution of the fading memory postulate of the Coleman-Mizel theory by an appropriate and stronger finite memory postulate. However, in Sec.2 we will show that introducing such a postulate always reduces the history space of the Coleman-Mizel theory to that of an elastic material (cf.[15]). On the other hand, we succeded in finding such a modification of the Coleman-Mizel theory which admits the finite memory postulate without becoming trivial.

## THEORY

Each constitutive modelling is based on the idea that a reaction (a response) depends on an action (a stimulus), and moreover that dependence is continuous, or even smooth in a sense.
The continuity of that dependence postulate is used for two different purposes, which will be described as follows.
A: Take the above dependence, as an constitutive operator, given a priori, and use the continuity postulate for constructing a state space (e.g. the state of action histories) in a way to make the operator continuous.
B: Construct a state space or a history space together with its topology as an a priori notion and use the continuity as a postulate for the constitutive operator.

It turns out that the approach A was chosen in [7], while B is the concept of [1]-[3],[8] and several others. We choose here the approach B. According to [1], the construction of a history space should allow for the following postulates to hold.

(CM1) Histories can be statically prolonged and moreover static prolongations of equivalent histories are equivalent, too.
(CM2) All sections of histories (i.e.restrictions of histories) are histories (i.e. are in the space).
(CM3) Constant histories are in the space.

In fact, cancelling or even weakening of one those postulates is not acceptable. However, in [1]-[3] another postulate was implicitely assumed, namely:

(CM0) The history space is a Köthe-Toeplitz space $(B, ||\cdot||)$ with

$$B=V/\sim \ , \quad ||\phi|| = \nu(|\phi|)$$

$$\phi_1 \sim \phi_2 = \nu(|\phi_1-\phi_2|)=0, \text{ i.e.}$$

$\mu\{s\geq 0:\phi_1(s) \neq \phi_2(s)\} = 0$ with a function norm $\nu$ (cf.[3]), a Borel measure $\mu$ on $[0,\infty)$ and

$$V = \{\phi:\nu(|\phi|)<\infty\} .$$

In other words: it is assumed, that the history space is built up from all functions which make the function norm $\nu$ finite. But of course, the axioms of function norms (cf.[2],[3]) are by no means physical ones.
The unpleasant consequence of the postulate (CM0) (cf.Theorem 2.2 in [3]) is the following Theorem. A state space $(B,||\cdot||)$ satisfying (CM0)-(CM4) together with the additional finite memory postulate

(FM) $\exists\, \omega>0\ \forall\, \phi_1, \phi_2 \in B \quad \phi_1|[0,\omega]=\phi_2|[0,\omega] \Rightarrow \phi_1=\phi_2$

is isomorphic to that of an elastic material.

Proof. From (FM) it follows that the influence measure $\mu$ (cf.[1]) vanishes on $[\omega,\infty)$, i.e. $\mu([\omega,\infty))=0$. On the other hand (CM0)-(CM3) yield the implication $\mu((a,b))=0 \Rightarrow \mu((0,\infty))=0$ with arbitrary $0<a<b<\infty$. Hence $\mu((0,\infty))=0$, but that inturn implies that two histories are equivalent if their final values coincide

Now, we are canceling (CM0) assuming instead the following
(P) The history space is a Köthe-Toeplitz space $(B,||\cdot||)$ with

$$B=V/\sim \ , \ ||\phi||=\nu(|\phi|), \phi_1 \sim \phi_2 \Leftrightarrow \nu(|\phi_1-\phi_2|)=0$$

with a function norm as before and

$$V = \{\phi : \forall_{\sigma>0} \ \nu(|\phi_{(\sigma)}|)<\infty)\}$$

Here $\phi_{(\sigma)}$ denotes the $\sigma$-section of $\phi$.

Remark. The postulate (CM2) is a consequence of (P) hence we have only to consider (P),(CM1) and (CM3);in [15] it was proved that the state space of a material satisfying (P),(CM1),(CM3) together with (FM) can be represented by a space of histories defined on the finite interval $[0,\omega]$.

In fact, let $F$ be a Banach function space, the elements of which are $\mu'$- measurable functions on $[0,\infty]$, where the measure $\mu'$ is the restriction of $\mu$ to $[0,\omega]$. If $||\ ||'$ denotes the norm in $F$, then the first property of the finite history space is given by the following

(F1) If $\sigma_1,\sigma_2 \in F$ and $||\phi_1-\phi_2||=0$, then $\phi_1(0)=\phi_2(0)$ .

From this requirement follows that the measure $\mu'$ must possesses an atom s=0. To formulate the next property, let us notice that since now a function $\phi$ (or more precisely - an equivalent class) from $F$ can be regarded as a history of finite duration, the static prolongation $T^\sigma\phi$ of $\phi$ by the amount $\sigma$ is defined by

$$(T^\sigma\phi)(s)=\begin{cases}\phi(0), & \text{if } s\leqq\min(\sigma,\omega)\\ \phi(s-\sigma), & \text{if } \sigma<s\leqq\omega \ .\end{cases} \qquad (1)$$

Note that the formula for $T^\sigma$ does not have any sense for $\sigma>\omega$. However, for any $\sigma\leqq\omega$ this map should be well defined in $F$, and should be continuous, as its counterpart in the case of the infinite memory was (cf.[8]). Hence the next requirement will be:
(F2) For any $\sigma$ $[0,\omega]$ the map $T^\sigma$ defined by (1) is continuous as a map from $F$ into $F$.

The image of $\phi$ under the map $T^\sigma$ can be regarded as the result of a composition of an element from $F$ with a constant function $\phi(0)$ on $[0,\sigma]$. In order to be able to compose elements from $F$ with nonconstant functions, called processes, one introduces a class $\Pi$ of action-valued functions defined on the closed intervals of the type $[0,d]$, $d\geqq0$. That class is introduced in the way, which makes possible to prolong a finite 'history' by a process to get a new finite history. The properties of the prolongation are introduced by an additional postulate (cf. Postulate (F3) in [15])and are natural for the model of material with finite memory. Thanks to this the properties of the class $\Pi$ are similar to that required by Noll in his framework of a "a new mathematical theory of materials" [6].

EXAMPLE

For a material with finite memory the influence measure $\mu'$ has the properties:
- it possesses an atom at s=0,
- it is absolutely continuous on $(0,\infty)$,
- it vanishes on $(\omega,\infty)$.

Hence a linear constitutive functional on that history space has the form:

$$r(\phi) = K\phi(0) + \int_0^\infty k(s)\phi(s)ds,$$

where the kernel k(s) vanishes for $s>\omega$ and K is constant.
For practical use we choose $k(\cdot)$ to be a smooth function of an assumed form, e.g.

$$k(s)=\begin{cases}k(1-s^2), & s\in[0,1]\\ 0 & s\geqq1\end{cases} \quad \text{and} \quad \omega=1.$$

This implies $\dot{r}= K\dot{\phi} -\phi +2\int_0^1 s\phi(s)ds$

For free oscillation of a spring of such a material with k=1 and a mass M we obtain the equation of motion

$$\int_0^\omega k(s)u(t-s)ds + K\,u(t)+ M\frac{d^2u(t)}{dt^2} = 0$$

with $u = e^{\lambda t}$ we obtain the characteristic equation for $\lambda$

$$K + \int_0^\omega k(s)e^{-\lambda s}ds + M\lambda^2 = 0$$

This equation may be written in the form

$$\lambda = i\sqrt{\frac{K}{M} + \int_0^\omega \frac{k(s)}{M}e^{-\lambda s}ds}$$

and solved by simple iteration. The solution for varying M is plotted in Figure 1.

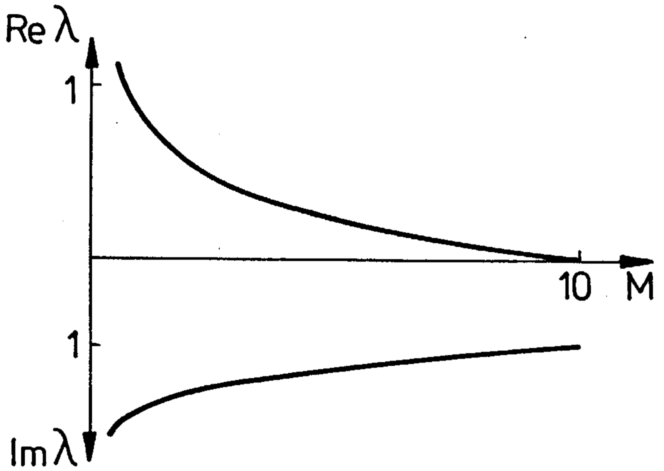


Fig.1

## FINAL REMARK

For 1-D structures with a finite memory constitutive law, especially for two layer vibrating beams with given elastic core and a damping layer with finite memory one reobtains the above characteristic equation. However, now the problem is coupled with the eigenvalue problem for the spacial problem. This together with the optimality problem for the distribution of the damping material is considered in the forthcoming paper [17].

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