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BRIEF NOTES

On the existence of materials with absolute memory

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IN A RECENT PAPER Frischmuth and Kosiński discussed a special case of materials with temporal memory, the so-called materials with absolute memory. It was stated that the memory of such materials must fade with time. This result would be meaningless if such materials did not exist. The aim of the present paper is to show by an example that the postulates of the mentioned paper allow for the existence of materials with absolute memory.

1. Introduction

ONE POSSIBILITY of describing the constitutive behaviour of material systems is to introduce a space of deformation histories $\mathscr{B}$ and to define then the stress as a function $r:\mathscr{B} \to \mathscr{S}$. Now, if one formally defines a space $\mathscr{B}$ and a function r from $\mathscr{B}$ to the space of stresses $\mathscr{S}$, it is doubtful whether one has described a "real" material system. Consequently, this gives rise to the problem of distinguishing the class of "real" materials within the class of formally possible descriptions. Therefore several postulates are introduced as the continuity of r and the existence of limit states in some sense.

One of those postulates, the relaxation property [1] of the history space $\mathscr{B}$ was shown to be improper for describing viscoplastic materials, and hence for this purpose other (weaker) postulates have to be found. The asymptotic rest property (AR) proposed in [2] was proved to have the same consequences as the relaxation property. On the other hand AR cannot be dropped in the case of materials with absolute memory, i.e. if

$$\forall \varphi \neq \psi \in \mathscr{B}, \quad \varphi(0) = \psi(0) \ \exists P \quad P(0) = \varphi(0) : r(\varphi \divideontimes P) \neq r(\psi \divideontimes P),$$

where $\varphi \divideontimes P$ denotes the continuation of the history φ by the process P. This implies that viscoplasticity is in contradiction with absolute memory. Consequently, materials with absolute memory must be semi-elastic. This result [2] is meaningful only if it can be shown that the class of materials with absolute memory is not empty. In [3] this was mentioned as an open question. The present paper solves it by giving a simple example of such a material.

2. Materials with absolute memory

In order to be possibly self-contained, let us briefly recall some notions which are used in the sequel.

First of all, we have a space $\mathscr{G}$ of configurations (or deformations) and a space $\mathscr{S}$ of stresses. Functions from the closed interval $[0, +\infty)$ to $\mathscr{G}$ are called deformation histories (or simpler histories). A history φ is defined by its final[1] value $\varphi(0)$ and by the past history $\varphi_r = \varphi|_{(0,\infty)}$ — as the restrictions of histories to the open interval $(0, \infty)$ are called. The space of all histories is called $\mathscr{B}$, the space of all past histories $\mathscr{B}_r$, hence $\mathscr{B} = \mathscr{G} \times \mathscr{B}_r$. By a process we denote a function P: Dom P = [0, dur P] $\to \mathscr{G}$ such that

$$\varphi \in \mathscr{B} \wedge \varphi(0) = P(0) \Rightarrow \varphi \ast P \in \mathscr{B}$$

with

$$(\varphi \ast P)(s) = \begin{cases} P(\operatorname{dur} P - s) & \text{for} \quad s \in [0, \operatorname{dur} P], \\ \varphi(s - \operatorname{dur} P) & \text{for} \quad s \in [\operatorname{dur} P, +\infty). \end{cases}$$

The history $\varphi \ast P$ is called the continuation of φ by P.

DEFINITION 1. *A collection* $(\mathscr{G}, \mathscr{S}, \mathscr{B}, r)$, $r: \mathscr{B} \to \mathscr{S}$, *is called a material with memory.*

The response function r defines an equivalence relation in $\mathscr{B}$ by (cf. [4])

$$\varphi \sim \psi \text{ iff } \varphi(0) = \psi(0) \wedge \forall P,\ P(0) = \varphi(0) \Rightarrow r(\varphi \ast P) = r(\psi \ast P).$$

The classes of this relation are called the states of the material. Of course, large states mean a "bad memory", fine partitions mean "good memory"". If the states are single histories, the memory is called absolute.

DEFINITION 2. *A material with memory is called a material with absolute memory if*

$$\varphi \sim \psi \Rightarrow \varphi = \psi.$$

3. Examples

In most cases the power of the set $\mathscr{B}$ equals that of $\mathscr{S}$ so that one can find a function $r: \mathscr{B} \to \mathscr{S}$ such that even

$$r(\varphi) = r(\psi) \Rightarrow \varphi = \psi$$

which ensures the above implication[2]. Unfortunately, such a response function is in general noncontinuous and because of that of no interest.

Another example is given by

$$\sigma(\varphi) = \begin{cases} 0 & \text{if} \quad \varphi \text{ is constant} \\ \sup\{s: \varphi(s) = \varphi(0)\} & \text{else} \end{cases}$$

$$r(\varphi) = \varphi(2\sigma(\varphi)).$$

We assume that $\mathscr{S} = \mathscr{G}$ is a subspace of a normed vector space. It is easy to see that for two different histories φ and ψ, say $\varphi(s_0) \neq \psi(s_0)$, the "test" P_{s_0}

$$P_{s_0} = \begin{cases} T & \text{on} \quad [0, 1], \\ C & \text{on} \quad [1, s_0 + 2] \end{cases}$$

(1) The argument $s \in [0, \infty)$ of a history is called the elapsed time. Consequently, we call $\varphi(0)$ a final value. It is the present value of the configuration.

(2) $\varphi \sim \psi \Rightarrow r(\varphi) = r(\psi) \Rightarrow \varphi = \psi$, because for dur $P = 0$ and $P(0) = \varphi(0) = \psi(0)$ we have $\varphi \ast P = \varphi$, $\psi \ast P = \psi$.

with

$$T(s) \neq T(1) \quad \text{for} \quad s < 1 \quad \text{and} \quad C \equiv T(1)$$

yields

$$r(\varphi \ast P_{s_0}) = \varphi(s_0) \neq \psi(s_0) = r(\psi \ast P s_0).$$

This implies absolute memory.

Now, if $\mathscr{B}$ is a subspace of a normed space with stronger topology than that of uniform convergence, we can conclude for constant histories:

$$\|\varphi_n - \varphi\|_{\mathscr{B}} < \delta \Rightarrow |r(\varphi_n) - r(\varphi)| = |\varphi_n(\sigma(\varphi_n)) - \varphi(0)| = |\varphi_n(\sigma(\varphi_n)) - \varphi(\sigma(\varphi_n))| < \varepsilon.$$

Consequently, r is continuous in all constants.

But in this example there are still some serious drawbacks so that we have to look for further examples. Usually histories being equal μ — a.e. with respect to some influence-measure μ are identified, and under the assumptions of [1] μ has no atoms but $s = 0$. For such history spaces the above definitions lose their sense. Furthermore it would be desirable to define a response function r which is continuous in its whole domain, not only in constants. To this end let us turn to our last example. For the sake of simplicity we put $\mathscr{G} = \mathscr{S} = \mathsf{R}$ and define the spaces of histories and past histories by

$$\mathscr{B} = \{\varphi : \mathsf{R}^+ \to \mathsf{R}, \int_0^\infty \varphi^2(t) h(t) dt < +\infty\},$$

$$\mathscr{B}_r = \{\varphi_r : \mathsf{R}^{++} \to \mathsf{R}, \int_0^\infty \varphi^2(t) h(t) dt < +\infty\}.$$

The oblivator $h : \mathsf{R}^+ \to \mathsf{R}^{++}$ is a continuous monotonically decreasing to zero function with $\int_0^\infty h(t) dt =: M < +\infty$. Letting

$$(\varphi, \psi)_{\mathscr{B}} = \varphi(0)\psi(0) + \int_0^\infty \varphi(s)\psi(s)\varphi(s)h(s) ds$$

and

$$(\varphi_r, \psi_r)_{\mathscr{B}_r} = \int_0^\infty \psi_r(s)\varphi_r(s)h(s) ds$$

we make $\mathscr{B}$ and $\mathscr{B}_r$ Hilbert spaces with $\mathscr{B} = \mathscr{B}_r \oplus \mathsf{R}$. By mapping $\mathscr{B}_r$ isometrically onto $L_2([0, \pi])$ via the substitution

$$x = \frac{\pi}{M} \int_0^t h(\tau) d\tau, \quad f(x) = \sqrt{\frac{M}{\pi}}\, \varphi(t),$$

we obtain the following:

LEMMA

$$\varphi_r \equiv 0 \quad \text{iff } \forall n \in \mathsf{N}\, (\varphi_r, k(n_i))_{\mathscr{B}_r} = 0,$$

where

$$k(n,t) = \sin\frac{n\pi}{M}\int_0^t h(\tau)\,d\tau.$$

Now we put

$$r(\varphi) := \varphi(0) + \big(\varphi_r, k(n(\varphi_r), \cdot)\big)_{\mathscr{B}_r},$$

$$n(\varphi_r) := \int_0^1 |\varphi_r(\tau)|h(\tau)\,d\tau.$$

THEOREM. *$(\mathscr{G}, \mathscr{S}, \mathscr{B}, r)$ describes a material with absolute memory.*

Proof. Assume that $\varphi(0) \neq \psi(0)$, but $\varphi_r \neq \psi_r$. We denote

$$\tilde{\varphi}_r(t) = \begin{cases} 0 & t\in[0,1] \\ \varphi_r(t-1) & t\in(1,\infty)\end{cases}, \quad \tilde{\psi}_r(t) = \begin{cases} 0 & t\in[0,1]^{(3)} \\ \psi_r(t-1) & t\in(1,\infty)\end{cases}.$$

Hence

$$\tilde{\varphi}_r, \tilde{\psi}_r \in \mathscr{B}_r, \quad \tilde{\varphi}_r \neq \tilde{\psi}_r.$$

By the Lemma there exists a number $n \in \mathrm{N}$ such that

$$(\tilde{\varphi}_r - \tilde{\psi}_r, k(n, \cdot))_{\mathscr{B}_r} \neq 0.$$

Now there exists a process P with $P(0) = \varphi(0) = \psi(0)$, dur $P = 1$ and

$$\int_0^1 P(1-\tau)h(\tau)\,d\tau = n.$$

Consequently,

$$r(\varphi \ast P) - r(\psi \ast P) = P(1) + \big((\varphi\ast P)_r, k(n, \cdot)\big)_{\mathscr{B}_r} - \Big(P(1) + \big((\psi\ast P)_r, k(n, \cdot)\big)_{\mathscr{B}_r}\Big)$$
$$= (\tilde{\varphi}_r - \tilde{\psi}_r, k(n, \cdot))_{\mathscr{B}_r} \neq 0.$$

This implies $\varphi \neq \psi \Rightarrow$ not true $(\varphi \sim \psi)$ or $\varphi \sim \psi \Rightarrow \varphi = \psi$ Q.E.D.

The above response function r is continuous. The proof is an immediate consequence of the chain rule and the estimation

$$\int_0^1 |\varphi(\tau)|h(\tau)\,d\tau \leqslant \int_0^\infty |\varphi(\tau)|h(\tau)\,d\tau \leqslant \Big(\int_0^\infty \varphi^2(\tau)h(\tau)\,d\tau\Big)^{1/2}\cdot\Big(\int_0^\infty h(\tau)\,d\tau\Big)^{1/2} \leqslant \sqrt{M}\,\|\varphi\|_{\mathscr{B}}.$$

REMARK. A generalization is easily obtained since the crucial condition for the above construction is just that there should exist a continuous function on R^+ with values in the dual space $\mathscr{B}_r^*$ such that $\bigcap_{t\in\mathrm{R}^+}\{\varphi\in\mathscr{B}_r : (\varphi, f(t)) = 0\} = \{0\}$.

4. Final comments

In this paper we were only interested in constructing an example of a continuous material with absolute memory. No other aspects of material theories have been con-

(3) $\tilde{\varphi}$, $\tilde{\psi}$ are the reduced continuations (by the amount 1) of φ_r and ψ_r, respectively (cf. [1]).

sidered. As regards the postulates of [2] we find all of them satisfied. (Of course, those postulates are only necessary conditions for a material to be real). Note that for the space $\mathscr{B}_r$ the relaxation property holds so that our example is in accordance with the mentioned statements of [2]. In the previous examples this was not the case since essential assumptions of that paper have not been fulfilled there.

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