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The asymptotic rest property for materials with memory

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IN ORDER to model viscoplasticity the Coleman–Mizel theory of materials with memory is modified by rejecting its two assumptions which introduced the fading character of the memory, namely the separability of the history space and the relaxation property of the norm. Remaining within the framework of the new Noll's theory of materials, the so-called asymptotic rest property (AR) is introduced, which ensures Noll's relaxation axiom. Unfortunately this property imposes the same restrictions as the previous one. From the theorem proved in the paper one infers that absolute memory is always fading and a viscoplastic memory is by no means absolute.

W celu opisu lepkoplastyczności dokonuje się modyfikacji teorii Colemana–Mizela materiałów z pamięcią przez odrzucenie jej dwóch założeń, które wprowadzały zanikający charakter pamięci, mianowicie ośrodkowości przestrzeni historii oraz własności relaksacji dla normy. Pozostając w ramach nowej teorii Nolla wprowadza się własności AR, która zapewnia aksjomat relaskacyjny Nolla. Niestety okazuje się, że wprowadzona własność narzuca takie same ograniczenia jak w teorii oryginalnej. Z udowodnionego twierdzenia wynika, że absolutna pamięć jest zawsze zanikająca oraz że pamięć materiałów lepkoplastycznych nie jest w żadnym wypadku absolutna.

С целью описания вязкопластичности проводится модификация теории Колемана–Мизеля материалов с памятью путем пренебрежения ее двумя предположениями, которые вводили исчезающий характер памяти, именно сепарабельности пространства истории и свойства релаксации для нормы. Оставляясь в рамках новой теории Нолла, вводится свойство АП, которое обеспечивает релаксационную аксиому Нолла. К сожалению оказывается, что введенное свойство навязывает те же самые ограничения, что и в оригинальной теории. Из доказанной теоремы следует, что абсолютная память является всегда исчезающей и что память вязкопластических материалов в никаком случае не является абсолютной.

1. Introduction

IN THE SERIES of papers [1–3] COLEMAN and MIZEL developed the general theory of materials with fading memory. This theory rests upon a Banach function space of the Köthe–Toeplitz type on which continuous constitutive operators are defined. The main part of the paper [3] deals with the restrictions imposed on the theory of materials with memory by four postulates, called here CM1–CM4, and by the relaxation property RP. The latter, as the main concept of the Coleman–Mizel theory, ensures Noll's relaxation axiom as well as semi-elasticity of the corresponding material element [4]. So it is quite natural to apply that theory in viscoelasticity. One can expect that in terms of response operators defined on a history space also a description of viscoplastic materials should be possible. The main aim of this paper is to examine once again the structure of the Coleman–Mizel theory in order to answer the question: Is it possible to model viscoplasticity by an analogous theory but with only slightly modified postulates?

Modifying the theory of materials with memory, we want to remain within the framework of Noll's definition of body and material elements. So we have to admit CM1 and CM2 unchanged and to substitute RP by a weaker assumption, which still ensures Noll's relaxation axiom. Such an assumption was first proposed by W. KOSIŃSKI and K. C. VALANIS in their manuscript [5] and was next called the "asymptotic rest property", AR in [6].

As to CM3, the separability of the history space, it should be recalled (cf [3]) that even without RP it introduces a fading character of the memory. Indeed, the postulate in question implies that Lebesque's theorem on dominated convergence holds and that tame-functions are dense in $\mathscr{B}$. These facts in turn can be used to illustrate the contradiction between the theory and real viscoplastic behaviour. Thus, in contrast to [5], we reject this postulate as being too restrictive. Our main theorem now asserts that AR is equivalent to the relaxation property together with the postulate CM4 of the existence of constant functions in the history space, under the assumption that CM1 and CM2 hold. This proposition was already obtained in [5], but using additionally CM3.

Hence it can be concluded that AR is only apparently weaker than RP and imposes the same restrictions. As a consequence we can state that the AR-property is improper in the theory of viscoplastic materials with temporal memory.

It should be pointed out that in the case of absolute memory AR is equivalent to Noll's axiom V about relaxation since in this case each state is represented by one history only.

So, if a constitutive operator of this type is considered, AR cannot be dropped. We infer by our theorem that absolute memory is always fading, and a viscoplastic memory is by no means absolute.

Comparing our theorem 2 with the theorem 1 proved for a more general constitutive theory in [6] we can see that the former theorem is stronger. This is so because of the special form of the evolution operators and a certain feature of their domain. Namely the convergence in norm of a sequence implies the pointwise (μ — a.e.) convergence of some subsequence to the same limit. It is the property that is the main tool in all proofs of the implication AR $\Rightarrow$ RP.

By our considerations it is impossible to discuss Noll's axiom V for a viscoplastic material theory until the constitutive operators and, consequently, the state space as a quotient set [4] are specified. Thus it is not very promising to create general theories of viscoplastic memory in an analogous way as the Coleman–Mizel theory was done, i.e. to generate constitutive operators by assuming an appropriate topology. In order to study the topology of the state and history spaces one should rather in advance assume some more properties of the constitutive operators than merely continuity. At least it should be known which histories are or are not distinguished by the memory of the material.

2. The space of histories

In order to keep the present paper sufficiently self-contained, let us recall in this section some of the most important facts from the Coleman–Mizel theory.

Following these authors, we introduce in the set of histories the structure of a Banach space. Let us first consider the set $\mathscr{S}$ of all functions ϕ mapping $[0, \infty)$ into itself, measurable with respect to a nontrivial, Σ — finite, positive, regular Borel measure, i.e. the so-called influence measure μ. A function ν defined on $\mathscr{S}$ will be called a nontrivial function norm with the sequential Fatou property if for all $\phi, \phi_i \in \mathscr{S}$

i) $0 \leqslant \nu(\phi) \leqslant \infty$ and $\nu(\phi) = 0$ if and only $\phi \overset{\circ}{=} 0$ (1),

ii) $\nu(\phi_1 + \phi_2) \leqslant \nu(\phi_1) + \nu(\phi_2)$ and $\nu(a\phi) = a\nu(\phi)$ for all numbers $a \geqslant 0$,

iii) if $\phi_1 \overset{\circ}{\leqslant} \phi_2$, then $\nu(\phi_1) \leqslant \nu(\phi_2)$,

iv) there is at least one function ψ in $\mathscr{S}$ with $0 < \nu(\psi) < \infty$,

v) if $\psi, \phi_1, \phi_2 \ldots$ are in $\mathscr{S}$ and if $\phi_n \nearrow \psi$ pointwise μ — a.e., then $\nu(\phi_n) \nearrow \nu(\psi)$ (Fatou property).

Let V be a nontrivial, separable, real Banach space with the norm $|\cdot|$, and let $\bar{\mathscr{V}}$ be the set of μ-measurable functions mapping $[0, \infty)$ into V. Now we define $\mathscr{V}$ by the relation

$$\mathscr{V} := \{\phi \in \bar{\mathscr{V}} : \nu(|\phi|) < \infty\}$$

and the semi-norm $||\cdot||$ on $\mathscr{V}$ by

$$||\phi|| := \nu(|\phi|) \quad \text{for all} \quad \phi \in \mathscr{V}.$$

Identifying two functions whenever the semi-norm of their difference vanishes, we obtain a normed function space $(\mathscr{B}, ||\cdot||)$.

The following results, known from the theory of normed Köthe spaces (cf. ZAANEN [7]), are essential for our considerations.

LEMMA 0. a) If ν is a function norm with the sequential Fatou property, then it has the Riesz–Fischer property, i.e.

vi) If $\{\phi_n\}_0^\infty \subset \mathscr{S}$ and $\sum_{n=0}^{\infty} \nu(\phi_n) < \infty$, then $\nu\left(\sum_{n=0}^{\infty} \phi_n\right) < \infty$;

b) if ν is a function norm with the Riesz–Fischer property, then each Cauchy sequence in $\mathscr{B}$ contains a subsequence which converges pointwise μ — a.e. to the limit in norm of the whole sequence (p. 445), and

c) the space $\mathscr{B}$ is complete.

The functions ϕ in $\mathscr{V}$ are called histories, their independent variable is usually denoted by s and is called the elapsed time. The value $\phi(0)$ is the present value of ϕ and the past values $\phi(s)$ are those for which $0 < s < \infty$. Consequently the restriction of a function to $(0, \infty)$ is called the past history of ϕ and is denoted by ϕ_r. We can define a space and a semi-norm by

$$\mathscr{V}_r = \{\phi_r : \bigvee \phi \in \mathscr{V},\ \phi_r = \phi|_{(0,\infty)}\},$$

$$||\phi_r||_r = ||\phi\chi_{(0,\infty)}|| = \nu(|\phi\chi_{(0,\infty)}|)$$

and obtain the space $\mathscr{B}_r$ of past histories by calling the same those past histories ϕ_r, ψ_r for which $||\phi_r - \psi_r||_r = 0$. Observe that Lmma 0 applies to $\mathscr{B}_r$ as well.

The domain of definition of a continuous constitutive operator r is used to be a cone $\mathscr{D} \subset \mathscr{B}$, i.e.

$$\mathrm{r}: \mathscr{B} \supset \mathscr{D} \to \mathscr{R},$$

(1) A superposed $\circ$ indicates that the given relation holds pointwise μ — a.e.

where $\mathscr{R}$ is often a finite-dimensional vector space. At the end of this section let us cite the first two postulates admitted in [3] and some of their consequences.

If ϕ is a function in $\tilde{\mathscr{V}}$, then we define

$$\phi^{(\sigma)}(s) = \begin{cases} \phi(0), & s \in [0, \sigma], \\ \phi(s-\sigma), & s \in [\sigma, \infty), \end{cases}$$
$$\phi_{(\sigma)}(s) = \phi(s+\sigma), \quad s \in [0, \infty).$$

POSTULATE CM1. If ϕ is in $\mathscr{V}$, then $\phi^{(\sigma)}$ is in $\mathscr{V}$ for all $\sigma \geqslant 0$. Furthermore, if ϕ and ψ are in $\mathscr{V}$ and $||\phi-\psi|| = 0$, then $||\phi^{(\sigma)}-\psi^{(\sigma)}|| = 0$ for all $\sigma \geqslant 0$.

COROLLARY 1. If we put

$$E^{\sigma}\phi := \phi^{(\sigma)},$$

then E^{σ} is a well-defined operator on $\mathscr{B}$ with values in $\mathscr{B}$. We will call it the static continuation by the amount σ.

POSTULATE CM2. If ϕ is in $\mathscr{V}$, then so are also all functions $\phi^{(\sigma)}$, $\sigma \geqslant 0$. As consequences of CM1–CM2 one can receive among others the following results [1–3]:

LEMMA 1. a) The measure μ must have an atom at $s = 0$ and be absolutely continuous on R^{++} with respect to the Lebesque measure λ. Furthermore either $\mu(R^{++}) = 0$ or λ is absolutely continuous on R^{++} with respect to μ. b) The space $\mathscr{B}$ is algebraically and topologically the direct sum of V and $\mathscr{B}_r$, $\mathscr{B} = V\oplus\mathscr{B}_r$ and the norm $||\cdot||$ is equivalent to $||\cdot||'$ defined by

$$||\phi||' := |\phi(0)|+||\phi_r||_r.$$

Hence it follows that the convergence in $\mathscr{B}$ is equivalent to the simultaneous convergence in $\mathscr{B}_r$ and V of the past histories and present values, correspondingly.

3. Relaxation property and asymptotic rest property

In the mathematical theory of materials developed by Coleman and Mizel the fading memory has been introduced through the postulate CM3 of separability of $\mathscr{B}$ and the relaxation property, bearing in mind the continuity of the response functional. In the present paper we take as the definition of the relaxation property the following relation:

$$\lim_{\sigma\to\infty} ||E^{\sigma}\phi-\phi(0)^{\dagger}|| = 0 \quad \text{for each} \quad \phi \in \mathscr{B}. \qquad \text{(RP)}$$

As announced in the introduction we try now to weaken RP, having CM3 rejected altogether. Our idea is that the limit $\lim_{\sigma\to\infty} E^{\sigma}\phi$ should exist for all ϕ and be allowed to depend on the past history ϕ_r of ϕ. This is the way we want to introduce the necessary, for visco-plastic modelling, nonfading (permanent) memory. Note that Noll's axiom V implies $\lim_{\sigma\to\infty} \mathrm{r}(E^{\sigma}\phi)$ to exist for each $\phi \in \mathscr{B}$. Consequently our assumption is the only possibility to get an "universal theory" which, regardless how r is defined, always yields a material description where the existence of those limits is ensured.

Definition. We say the norm $||\cdot||$ and $\mathscr{B}$ possess the asymptotic rest property if

$$\lim_{\sigma\to\infty} E^{\sigma}\phi \quad \text{exists in } \mathscr{B} \quad \text{for each} \quad \phi \in \mathscr{B}. \qquad \text{(AR)}$$

In order to formulate the main theorem we recall that the last Coleman–Mizel postulate, CM4, demanded the existence of nontrivial constant functions in $\mathscr{V}_\gamma(\mathscr{V})$, i.e. the inequality $\nu(\chi_{(0,\infty)}) < \infty$ must be satisfied. Now we are able to prove the following main result.

THEOREM 1. *The asymptotic rest property implies the relaxation property and the Coleman–Mizel postulate* CM4.

Proof. By AR and point b of Lemma 0 we infer for an arbitrary $\phi \in \mathscr{B}$ the existence of a sequence $\{\sigma_n\}_0^\infty$, $\sigma_n \underset{n\to\infty}{\to} \infty$ such that $E^{\sigma_n}\phi$ converges pointwise μ — a.e. to $\lim\limits_{\sigma\to\infty} E^\sigma\phi$. On the other hand, for σ_n tending to infinity the sequence $\{E^{\sigma_n}\phi\}$ converges pointwise on $[0, \infty)$ to $\phi(0)^\dagger$. Thus $\lim\limits_{\sigma\to\infty} E^\sigma\phi \overset{\circ}{=} \phi(0)^\dagger$. Now it is sufficient to take a $\phi \in \mathscr{B}$ with $\phi(0) \neq 0 \in V$ to conclude $\nu(\chi_{[0,\infty)}) < \infty$ and hence all constant functions are in $\mathscr{V}$ Q.E.D.

Since the relaxation property trivially implies AR, we have the following:

COROLLARY. The relaxation property and the asymptotic rest property are equivalent.

REMARK 1. If one wants to create a theory without CM4, then the original formulation of RP and a corresponding version AR* of the asymptotic rest property can be used and shown to be equivalent, independently whether CM4 occurs or not. In [3] COLEMAN and MIZEL formulated the relaxation property in terms of the condition

$$\lim_{\sigma\to\infty} \|T^{(\sigma)}\phi_r\| = 0, \qquad \forall \phi_r \in \mathscr{B}_r. \qquad \text{(RP*)}$$

Here $\{T^{(\sigma)}, \sigma \geqslant 0\}$ is the family of σ — reduced continuations defined by

$$(T^{(\sigma)}\phi_r)(s) = \begin{cases} 0, & s \in (0, \sigma], \\ \phi(s-\sigma), & s \in (\sigma, \infty). \end{cases}$$

The corresponding version of the asymptotic rest property is then

$$\lim_{\sigma\to\infty} T^{(\sigma)}\phi_r \quad \text{exists for each} \quad \phi_r \in \mathscr{B}_r. \qquad \text{(AR*)}$$

A repetition of the proof of Theorem 1, *mutatis mutandis*, gives the following result.

LEMMA 2. The star–versions of the asymptotic rest and relaxation property are equivalent.

REMARK 2. Using the closed graph theorem one can easily prove that for each σ the operators $T^{(\sigma)}$ and E^σ are linear bounded operators. An application of this fact can be used in another proof of Theorem 1.

Proof of Theorem 1 (second version of AR $\Rightarrow$ RP). The continuity of E^τ implies (²) that $E^\tau(\lim\limits_{\sigma\to\infty} E^\sigma\phi) = \lim\limits_{\sigma\to\infty} E^\sigma\phi$ for each $\phi \in \mathscr{B}$ and each $\tau \in R^+$. Let us denote $\lim\limits_{\sigma\to\infty} E^\sigma\phi =: \phi^f$ and consider the set $A := \{s > 0 : \phi^f(s) \neq \phi^f(0)\}$. In order to show that $\mu(A) = 0$ it is sufficient to show that $\mu(A \cap [0, n]) = 0$ for any natural n. Indeed, $\phi^f = E^n\phi^f$ and consequently $\phi^f(s) = (E^n\phi^f)(s) \equiv \phi^f(0)$ for $s \in [0, n]$. Thus $\mu(A \cap [0, n]) = 0$ but this in turn proves

$$\mu\Big(\bigcup_{n=1}^{\infty} (A \cap [0, n])\Big) = \mu(A) = 0.$$

(²) In this version point b of Lemma 0 is used indirectly; it is essential in the proof of the continuity of the operators. Also in the original proof of Kosiński and Valanis the same fact is used. For the proof of the continuity of $T^{(\sigma)}$ cf. [1], the same argument applies to E^σ.

Hence ϕ^f is a constant function. Now point b of Lemma 1 yields $\phi^f(0) = \phi(0)$. Since ϕ was arbitrary, we obtained the desired result. Analogously we can prove that $\lim_{\sigma \to \infty} T^{(\sigma)}\phi_r$ is represented by the zero function.

REMARK 3. In a previous paper the first author [6] examined a general approach to describing material bodies. It is easy to show that the present model is a particular case of that in [6]. The evolution operators $\{T_t\}$ introduced in [6] may be expressed by $\{E^\sigma\}$; the original postulates P1 and P2 from [6] formulated now would be nothing else than the continuity of each E^σ and AR, correspondingly. The thesis of our theorem is analogous to that of Theorem 1 of [6], but in the present case no assumptions about the asymptotic stability and continuity are necessary. Moreover, since here the evolution operators T_t have a known form with the particular strong properties, we obtain the asymptotic stability in the large as a result and at last the uniqueness of the equilibrium (relaxed) state corresponding to a fixed configuration.

4. Final comments

The idea of the Coleman–Mizel theory was to create a theory of materials without specifying the form of the response operator. In that theory any freeze is allowed and the state of the material reaches a limit for the duration of freeze tending to infinity, provided that r is merely continuous. The last statement is ensured by the relaxation property which, despite the fact that it is very restrictive, fortunately is not in contradiction with the (real) material behaviour called viscoelasticity. Specifying μ, ν and r one can correctly describe each real material of the concerned class to an arbitrary small error.

In viscoplaticity one again wants to construct a class of hereditary laws, consistent with the Noll's new theory and wide enough for practical and theoretical applications, but as it was shown in the previous section the matter is here more complicated. In the real viscoplastic behaviour there is a nontrivial dependence of the relaxed state on the past history under the actual configuration kept constant. Consequently the relaxation property is improper for viscoplastic modelling.

We infer that there is no condition which would simultaneously meet reality and ensure universally that regardless which continuous function r: $\mathscr{B} \to \mathscr{R}$ is considered, the state during relaxation processes reaches a limit (³).

So it seems to be natural to define in advance the state space, i.e. to introduce an equivalence relation in the space of histories. Each element of the resulting quotient space, i.e. a class of this relation, comprises such histories, which cannot be distinguished by observation of the reaction to processes [4] being a prolongation of the histories considered. Then the axiom V of Noll's theory can be discussed in terms of the quotient space.

Two realisations of such equivalence relations should be mentioned: time rescaling and internal variables.

At last let us consider a special case, namely that of one-element classes, i.e. when the equivalence relation is the identity and each state is represented by only one history.

(³) The fact that AR is a necessary condition of the latter property (the „universal" version of Noll's axiom V) can be proved by some standard arguments from general topology.

We then say that the material has absolute memory. In this case AR follows Noll's axiom V and we conclude that absolute and nonfading memory are incompatible. This statement can be interpreted as follows: the memory of each material looses information — either during relaxation in the fading memory case, or at once if different histories belonging to one class are undistinguishable.

Our results remain valid for each history space which satisfies point b of Lemma 0, as for example spaces embedded continuously in a Köthe–Toeplitz one.

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